Exchange Rates Modeling in Copulas

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Outline

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Motivations

- National economical, trade and financial situation will be reflected in the exchange rate, that is, exchange rate movements can see the country in a given period of the financial situation.
- The financial situation of each country must be among the dependence, but not know the extent of dependency.
- In the simple case, we do exchange rate research for the two developed countries.
- Using Copula function to describe the correlation of these two countries, and then write down their joint model, can predict the future of exchange rate movements for these two countries.

Outline

Introduction of Model Assumption

- **Marginal processes:** Stochastic model, Geometric Brownian motion.
  For $k = 1, 2$,
  $$dY_{kt} = \mu_k dt + \sigma_k dW_{kt}$$
  ✓ μ_k and σ_k are unknown parameters, W_{kt} is a Brownian motion.
  Discretize $\Rightarrow Y_{kt} - Y_{k,t-1} = \mu_k \Delta t + \sigma_k \sqrt{\Delta t} \varepsilon_{kt}$, where $\varepsilon_{kt} \sim N(0, 1)$.
  ✓ Note $Y_{kt} = \ln S_{kt} - \ln S_{k,t-1}$ and S_{kt} is the exchange rate at time t.
- **Dependence structure:** Copula function.
  Including $Normal$ copula, t copula, ...

Figure: Proposed Method 1

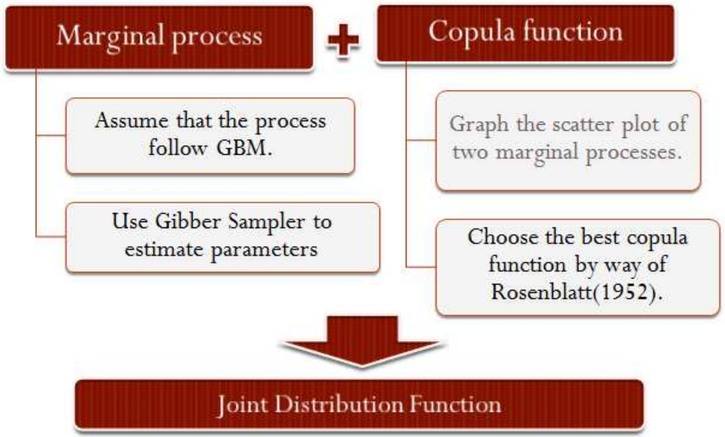
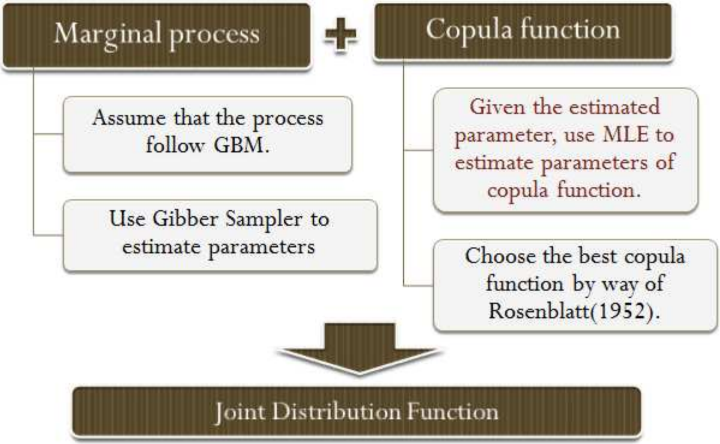


Figure: Proposed Method 2



Sklar's theorem in 2-dim case

Let $F(x, y)$ be a joint c.d.f with marginal c.d.f F_1 and F_2. There exists a copula C such that for all real (x, y),

$$F(x, y) = C(F_1(x), F_2(y)) \tag{1}$$

If F_1 and F_2 are continuous, then the copula is unique; otherwise, C is uniquely determined on (range of F_1) $\times$ (range of F_2). Conversely, if C is a copula, F_1 and F_2 are c.d.fs, then $F(x, y) = C(F_1(x), F_2(y))$ is a joint c.d.f with F_1 and F_2 as margins.

Introduction of Proposed Methods

- **Marginal processes:**
  We use Markov Chain Monte Carlo (MCMC) in Bayesian method.

✓ Priors: $\mu_k \sim Normal(\overline{\mu}_k, \tau_k^2) \perp \sigma_k^2 \sim IG(\alpha_k, \beta_k)$

✓ Likelihood function: $y_{kt}|y_{k,t-1} \sim N(y_{k,t-1} + \mu_k \Delta t, \sigma_k^2 \Delta t)$ for $t = 2, ..., T$ and $y_{k1} \sim N(\mu_k \Delta t, \sigma_k^2 \Delta t)$

$$f(\mathbf{y_k}|\mu_k, \sigma_k^2) = f(y_{k1}, ..., y_{kt}|\mu_k, \sigma_k^2) \tag{2}$$

$$= f(y_{kt}|y_{k1}, ..., y_{k,t-1}, \mu_k, \sigma_k^2) \cdots f(y_{k1}|\mu_k, \sigma_k^2) \tag{3}$$

$$= (2\pi\sigma_k^2 \Delta t)^{-\frac{T}{2}} exp\{-\frac{1}{2\sigma_k^2 \Delta t} \sum_{i=1}^{T} (y_{ki} - y_{k,i-1} - \mu_k \Delta t)^2\} \tag{4}$$

✓ Gibbs sampler:

$$\mu_k|\mathbf{y}_k,\sigma_k^2 \sim N(\frac{\tau_k^2\Delta t\sum_{i=1}^{T}(y_{ki}-y_{k,i-1})+\overline{\mu}_k\Delta t\sigma_k^2}{T(\Delta t)^2\tau_k^2+\Delta t\sigma_k^2},\frac{\tau_k^2\sigma_k^2\Delta t}{T(\Delta t)^2\tau_k^2+\Delta t\sigma_k^2})$$

$$\sigma_k^2|\mathbf{y}_k,\mu_k \sim IG(\frac{t}{2}+\alpha_k,[\frac{1}{2\Delta t}\sum_{i=1}^{T}(y_{ki}-y_{k,i-1}-\mu_k\Delta t)^2+\frac{1}{\beta_k}]^{-1})$$

- **Dependence structure:** Let $U = F(Y_{1t})$ and $V = F(Y_{2t})$

(i) Normal copula :

$$C^N(u,v;r) = \int_{-\infty}^{\Phi^{-1}(u)}\int_{-\infty}^{\Phi^{-1}(v)}\frac{1}{2\pi\sqrt{1-r^2}}exp\{-\frac{s^2-2rst+t^2}{2(1-r^2)}\}dsdt$$

(ii) t copula :

$$C^t(u,v;r,\nu) = \int_{-\infty}^{t_\nu^{-1}(u)}\int_{-\infty}^{t_\nu^{-1}(v)}\frac{1}{2\pi\sqrt{1-r^2}}\{1+\frac{s^2-2rst+t^2}{\nu(1-r^2)}\}^{\frac{-\nu+2}{2}}dsdt$$

✓ MLE:

Estimate the parameters in copula function taking the parameters estimated in marginal processes as given.

- **Joint Distribution Function:**
  For $k = 1, 2$, the estimated marginal processes, $d\hat{Y}_{kt} = \hat{\mu}_k dt + \hat{\sigma}_k dW_{kt}$. Then,
  $\hat{U} = \hat{F}_1(Y_{1t}) = \Phi_1(\hat{Y}_{1t})$ or $t_{1,\nu}(\hat{Y}_{1t})$ and
  $\hat{V} = \hat{F}_2(Y_{2t}) = \Phi_2(\hat{Y}_{2t})$ or $t_{2,\nu}(\hat{Y}_{2t})$.

  Hence, the joint distribution, $\hat{F}(y_{1t}, y_{2t})$, is as the following.

  $$\hat{F}(y_{1t}, y_{2t}) = C^N(\hat{u}, \hat{v}; \hat{r})$$
  $$= \int_{-\infty}^{y_{1t}} \int_{-\infty}^{y_{2t}} \frac{1}{2\pi\sqrt{1-\hat{r}^2}} exp\{-\frac{s^2-2\hat{r}st+t^2}{2(1-\hat{r}^2)}\} dsdt$$
  $$\hat{F}(y_{1t}, y_{2t}) = C^t(\hat{u}, \hat{v}; \hat{r}, \nu)$$
  $$= \int_{-\infty}^{y_{1t}} \int_{-\infty}^{y_{2t}} \frac{1}{2\pi\sqrt{1-\hat{r}^2}} \{1 + \frac{s^2-2\hat{r}st+t^2}{\nu(1-\hat{r}^2)}\}^{\frac{-\nu+2}{2}} dsdt$$

Outline

Data analysis

- Number of data: 534

Figure: Statistics for daily log exchange rates of USD and EUA after taking logarithm.

Exchange Rate	USD	EUA
Mean	3.471287	3.773706
Variance	0.001133057	0.004005752
Covariance	0.001039131	
Correlation	0.4877559	

Figure: The original USD data after taking logarithm.

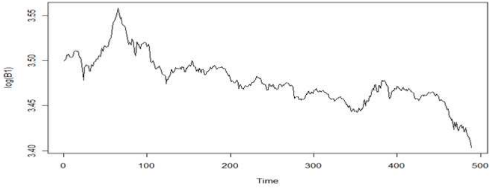


Figure: The original EUA data after taking logarithm.

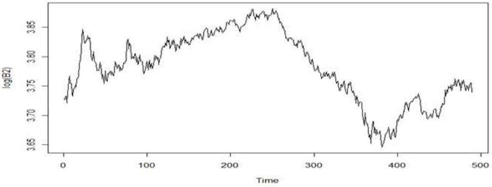


Figure: The original USD data after taking logarithm and differencing. i.e. Y_{1t}

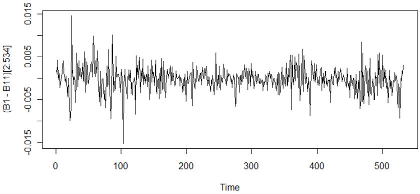


Figure: The original EUA data after taking logarithm and differencing. i.e. Y_{2t}

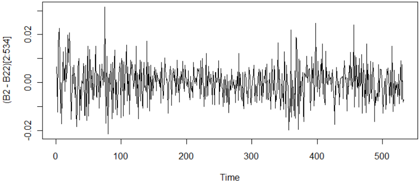


Figure: The empirical cdf of two marginal processes. Left figure is USD; right figure is EUA.

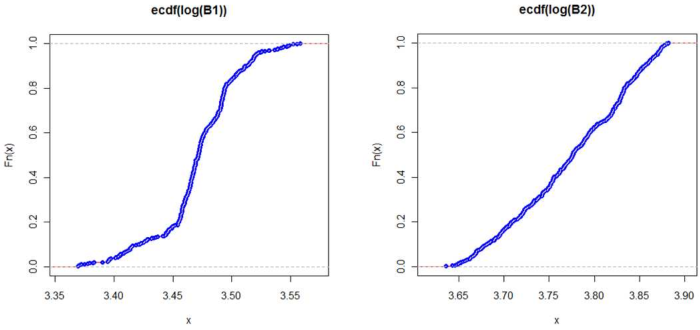
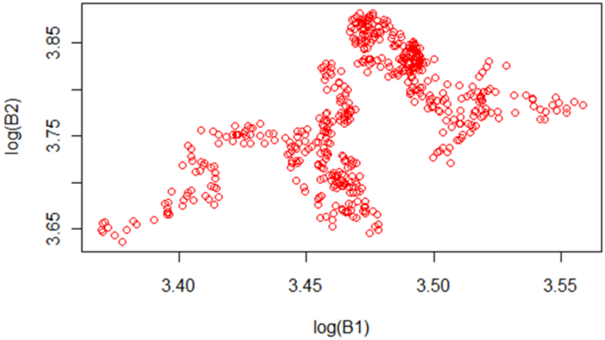


Figure: The scatter plot of two marginal empirical density function.$log(B1)$: USD; $log(B2)$: EUA.



- Priors:

✓ $\mu_1 \sim N(-0.0002281511, 0.000008423065)$ and $\mu_2 \sim N(-0.0001691731, 0.00006136552)$

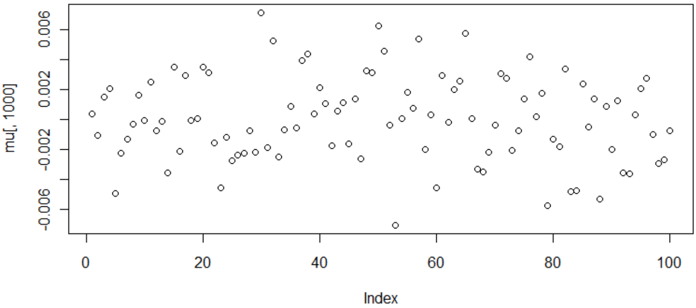
✓ $\sigma_1^2 \sim IG(3, 0.02)$ and $\sigma_1^2 \sim IG(3, 0.04)$

- Estimated parameters:

✓ $(\hat{\mu_1}, \hat{\sigma_1^2}) = (-0.0002408598, 0.09961398)$ and $(\hat{\mu_2}, \hat{\sigma_2^2}) = (-0.0001806745, 0.04986858)$

- mean of $SSR_1 = 13.42913$ and mean of $SSR_2 = 0.03303209$

Figure: The Gibbs Sampler.



[scale=0.35]Figure(USD)(2).eps
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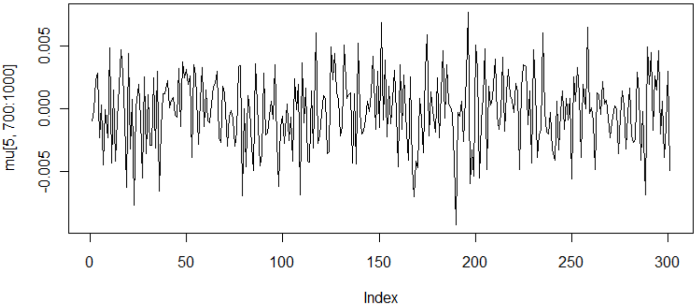
Figure: The Gibbs Sampler.

[scale=0.35]Figure(USD)(1).eps



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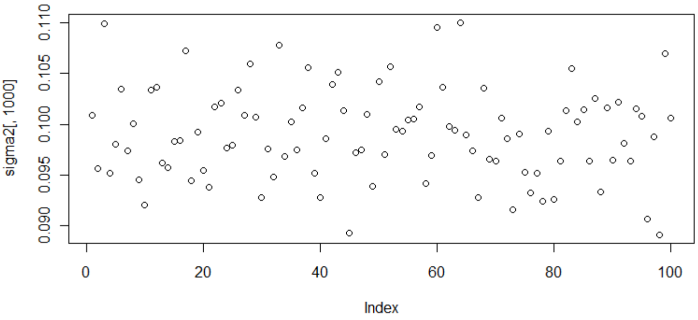

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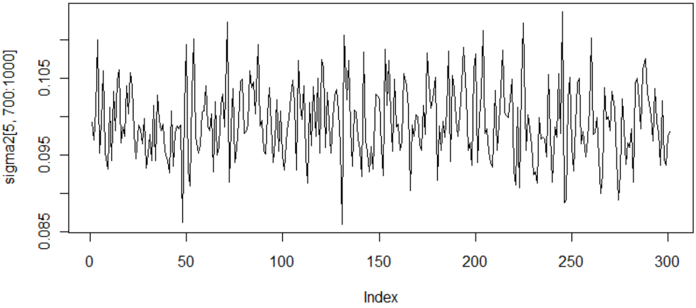

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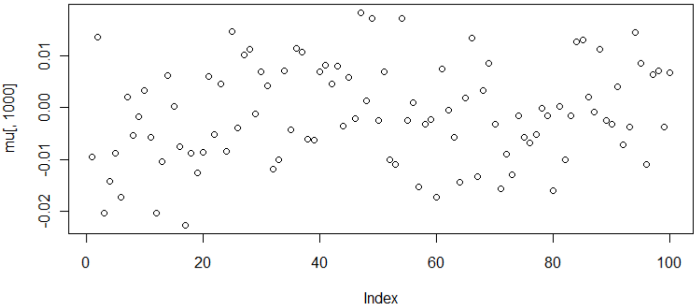

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Figure: The Gibbs Sampler.

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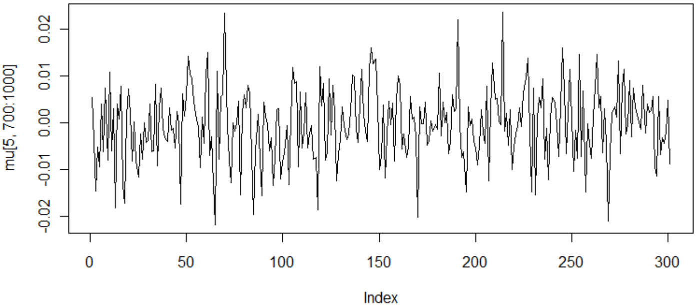

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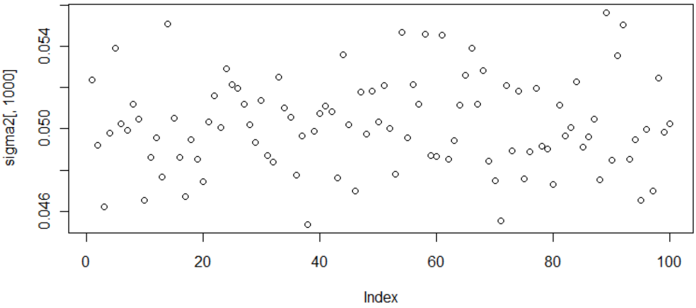

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Figure: The Gibbs Sampler.

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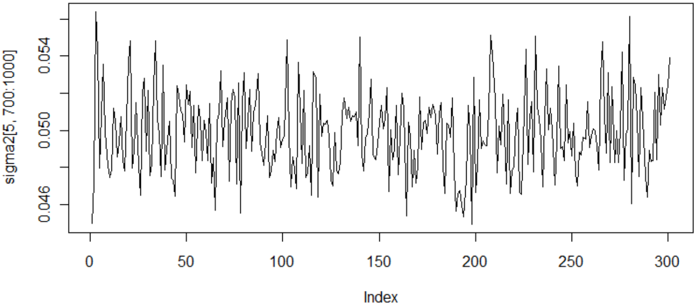

Figure: The estimated difference of log(exchange rate) of USD.

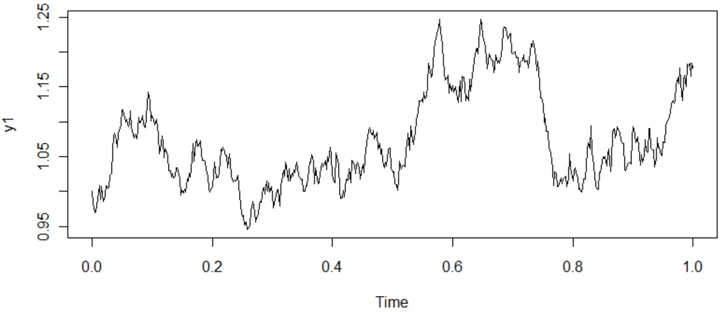

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Figure: The estimated difference of log(exchange rate) of USD.

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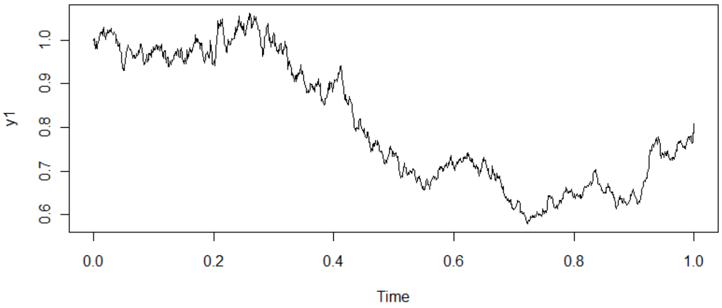

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Figure: The estimated difference of log(exchange rate) of USD.

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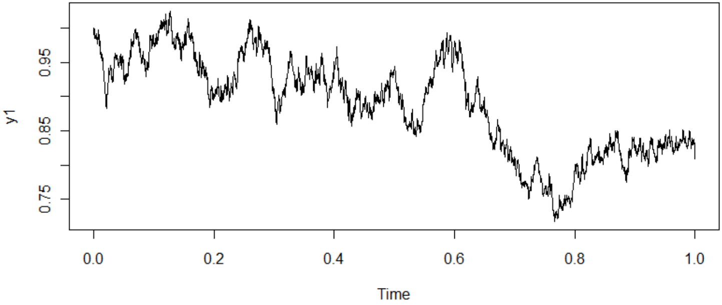

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Figure: The estimated difference of log(exchange rate) of USD.

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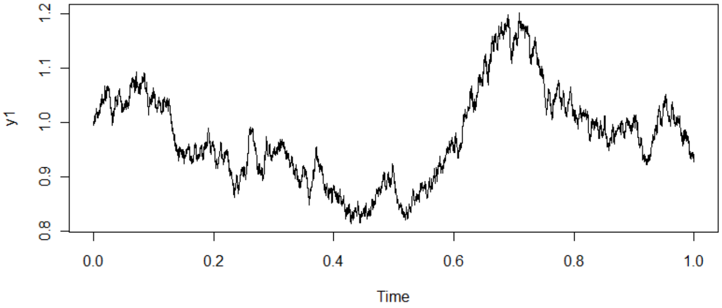

Figure: The estimated difference of log(exchange rate) of EUA.

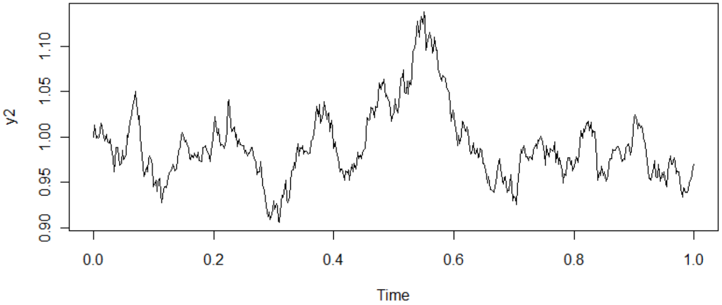

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Figure: The estimated difference of log(exchange rate) of EUA.

[scale=0.35]predict5.eps

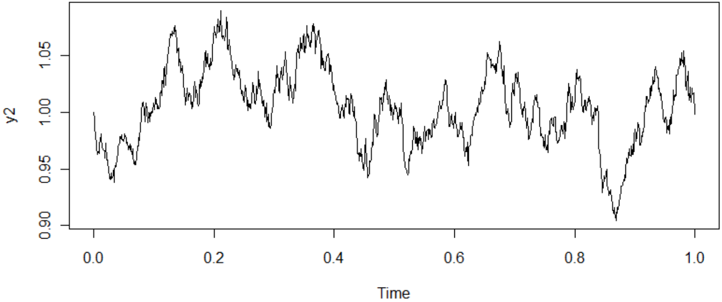

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Figure: The estimated difference of log(exchange rate) of EUA.

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[scale=0.35]predict6.eps

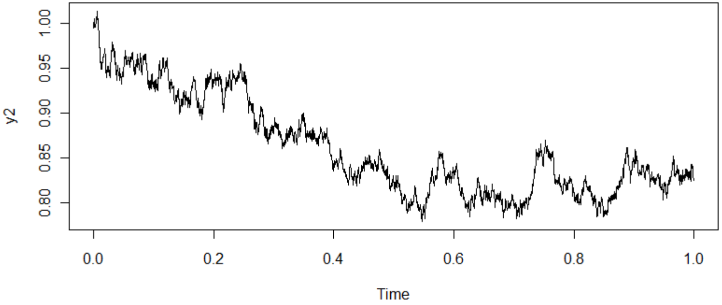

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Figure: The estimated difference of log(exchange rate) of EUA.

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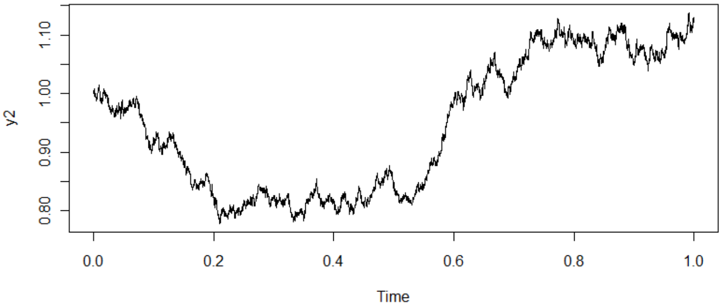

- Using MLE to estimate parameter of copula function as follows:
- Estimated parameter of Normal copula: $\hat{r} = 0.9865621$

$$C_{12}^{N}[F_1(y_{1t}), F_2(y_{2t}); r] = \frac{1}{2\pi\sqrt{1-r^2}} exp\{-\frac{y_{1t}^2 - 2ry_{1t}y_{2t} + y_{2t}^2}{2(1-r^2)}\}$$

- Estimated parameter of t copula: $\hat{r} = 0.9985747$

$$C_{12}^{t}[F_1(y_{1t}), F_2(y_{2t}); r] = \frac{1}{2\pi\sqrt{1-r^2}}\{1+\frac{y_{1t}^2 - 2ry_{1t}y_{2t} + y_{2t}^2}{\nu(1-r^2)}\}^{\frac{-\nu+2}{2}}$$

- Joint distribution function: let Y_{1t} be USD and Y_{2t} be EUA.

  $\hat{Y}_{1t} = -0.0002408598dt + 0.09961398dW_{1t}$ and
  $\hat{Y}_{2t} = -0.0001806745dt + 0.04986858dW_{2t}$

(i) Using Normal copula,
$\hat{U} = \hat{F}_1(Y_{1t}) = \Phi_1(\hat{Y}_{1t})$ and $\hat{V} = \hat{F}_2(Y_{2t}) = \Phi_2(\hat{Y}_{2t})$

✓ $\hat{F}(y_{1t}, y_{2t}|\hat{r} = 0.9865621) =$
$\int_{-\infty}^{y_{1t}} \int_{-\infty}^{y_{2t}} \frac{1}{2\pi\sqrt{1-\hat{r}^2}} exp\{-\frac{s^2-2\hat{r}st+t^2}{2(1-\hat{r}^2)}\}dsdt$

(ii) Using t copula,
$\hat{U} = \hat{F}_1(Y_{1t}) = t_{1,\nu}(\hat{Y}_{1t})$ and $\hat{V} = \hat{F}_2(Y_{2t}) = t_{2,\nu}(\hat{Y}_{2t})$.

✓ $\hat{F}(y_{1t}, y_{2t}|\hat{r} = 0.9865621) =$
$\int_{-\infty}^{y_{1t}} \int_{-\infty}^{y_{2t}} \frac{1}{2\pi\sqrt{1-\hat{r}^2}} \{1 + \frac{s^2-2\hat{r}st+t^2}{\nu(1-\hat{r}^2)}\}^{\frac{-\nu+2}{2}} dsdt$

Outline

Conclusion

- In the fitted model, we could predict the future exchange rates and see USD how to effect trend of EUA. According to the results, we can see the two exchange rates decrease simultaneously.
- It means that economical situation in the two regions are not well and influence each other probably .
- The two exchange rates seem to decline especially in the last months. However, there is no much information to judge the trend in the future.

Future Work

- We can extend (μ, σ^2) to (μ_t, σ_t^2) in marginal processes. That is, it is a better method to modelling as μ_t and σ_t^2 depend on time t.
- We can choose another copula functions to describe dependence structure like as Gumbel copula, Frank copula,..., and so on. Because there is no elliptical pattern in their scatter plot, we do not use elliptical copulas to link it.
- Besides, we should make a test for our copulas to choose a suitable copula.

Reference

- Shang Chan, Chiou (2006), Multivariate Continuous Time Models through Copula. (Doctoral dissertation)
- Nelsen, R. (1999): An Introduction to Copulas. Springer, New York.
- Paul Embrechts, Filip Lindskog, and Alexander McNeil (2001), Modelling Dependence with Copulas ans Applications to Risk Management.
- Xiaohong Chen, Yanqin Fan, and Andrew Patton (2004), Simple Tests for Models of Dependence between Multiple Financial Time Series, with Applications to U.S. Equity Returns and Exchange Rates. (working paper)

Thank You for Your Attention!!