**Quiz #1, Survival Analysis I, 2016 Spring**

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- **Not only answer but also derivations**

1. Let $T = e^Y$ be lifetime under the log-location-scale model with

$$\Pr(Y \geq y) = S_0\left(\frac{y-u}{b}\right).$$

+1 (i) Derive the p.d.f. of $T$ under $S_0(u) = 1 - \Phi(u)$, where $\Phi(x)$ is the c.d.f. of $N(0,1)$

$$f_0(u) = \frac{-d}{du} S_0(u) = -(-\phi(u)) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{u^2}{2}}$$

Let $U = \frac{Y-u}{b} \Rightarrow du = \frac{1}{b} dy$

$$f(y) = \frac{1}{b} \frac{1}{\sqrt{2\pi}} e^{-\frac{(y-u)^2}{2b^2}} \sim N(u, b^2)$$

Let $Y = \log T \Rightarrow dy = \frac{1}{t} dt$

$\therefore$ The p.d.f. of $T$ is $f(t) = \frac{1}{bt} \frac{1}{\sqrt{2\pi}} e^{-\frac{(\log t - u)^2}{2b^2}} \sim LN(u, b^2)$. #

+1 (ii) Derive the p.d.f. of $T$ under $S_0(u) = \exp(-e^u)$. Write the p.d.f. by parameters $\alpha = e^u$ and $\beta = 1/b$.

$$f_0(u) = \frac{-d}{du} S_0(u) = \exp(-e^u) \cdot e^u = \exp\{u - e^u\}.$$

Let $U = \frac{Y-u}{b} \Rightarrow du = \frac{1}{b} dy.$

$$f(y) = \frac{1}{b} \exp\left\{\frac{y-u}{b} - e^{\frac{y-u}{b}}\right\}.$$

Let $Y = \log T \Rightarrow dy = \frac{1}{t} dt.$

$$f(t) = \frac{1}{bt} \exp\left\{\frac{\log t - u}{b} - e^{\frac{\log t - u}{b}}\right\}.$$

$\left[\text{Let } V = \log s,\ dv = \frac{1}{s} ds,\ \begin{array}{c|c} s & t \to \infty \\ \hline v & \log t \to \infty \end{array}\right]$

$$S_T(t) = \int_t^{\infty} \frac{1}{bs} \exp\left\{\frac{\log s - u}{b} - e^{\frac{\log s - u}{b}}\right\} ds = \int_{\log t}^{\infty} \frac{1}{b} \exp\left\{\frac{v-u}{b} - e^{\frac{v-u}{b}}\right\} dv$$

$\left[\text{Let } W = \frac{v-u}{b},\ dw = \frac{1}{b} dv,\ \begin{array}{c|c} v & \log t \to \infty \\ \hline w & \frac{\log t - u}{b} \to \infty \end{array}\right]$

$$= \int_{\frac{\log t - u}{b}}^{\infty} \exp\{w - e^w\} dw = \int_{\frac{\log t - u}{b}}^{\infty} e^w e^{-e^w} dw = -e^{-e^w}\Big|_{\frac{\log t - u}{b}}^{\infty}$$

$$= e^{-e^{\frac{\log t - u}{b}}} = \exp\left\{-\left(\frac{e^y}{\alpha}\right)^{\beta}\right\} = \exp\left\{-\left(\frac{t}{e^u}\right)^{\frac{1}{b}}\right\}$$

$\left[\text{Let } y = \log t,\ u = \log \alpha,\ b = \frac{1}{\beta}\right]$

$\rightarrow$ It's the survival function of Weib($e^u$, $\frac{1}{b}$), $\alpha = e^u$

$$\therefore \text{ The p.d.f. of } T \text{ is } f(t) = \frac{\frac{1}{b}}{e^u} \cdot \left(\frac{t}{e^u}\right)^{\frac{1}{b}-1} \exp\left\{-\left(\frac{t}{e^u}\right)^{\frac{1}{b}}\right\}$$

$$= \frac{\beta}{\alpha} \cdot \left(\frac{t}{\alpha}\right)^{\beta-1} \exp\left\{-\left(\frac{t}{\alpha}\right)^{\beta}\right\} \sim \text{Weib}(\alpha = e^u, \beta = \tfrac{1}{b})$$

2. Let $T \sim LN(\mu,\sigma^2)$

+1 (i) Derive $E[T]$ and $Var[T]$.

Let $Y = \log T \sim N(\mu, \sigma^2)$

$$M_Y(t) = \exp\{\mu t + \frac{\sigma^2}{2}t^2\}$$

$$E[T] = E(e^Y) = M_Y(t=1) = \exp\{\mu + \frac{\sigma^2}{2}\} \#$$

$$Var[T] = E[T^2] - (E[T])^2 = E(e^{2Y}) - e^{2\mu+\sigma^2} = M_Y(t=2) - e^{2\mu+\sigma^2}$$

$$= e^{2\mu+2\sigma^2} - e^{2\mu+\sigma^2} = e^{2\mu+\sigma^2}(e^{\sigma^2}-1) \#$$

+1 (ii) Derive the hazard function $h(t)$ of $T$

$$f(t) = \frac{1}{\sigma t}\frac{1}{\sqrt{2\pi}} e^{-\frac{(\log t-\mu)^2}{2\sigma^2}} = \frac{1}{\sigma t}\phi\left(\frac{\log t-\mu}{\sigma}\right), \text{ where } \phi(x) \text{ is the p.d.f of } N(0,1).$$

$$F(t) = Pr(T \le t) = Pr(e^Y \le t) = Pr(Y \le \log t) = Pr\left(\frac{Y-\mu}{\sigma} \le \frac{\log t-\mu}{\sigma}\right) = \Phi\left(\frac{\log t-\mu}{\sigma}\right),$$

where $\Phi(x)$ is the c.d.f. of $N(0,1)$.

$$S(t) = 1 - \Phi\left(\frac{\log t-\mu}{\sigma}\right)$$

$$\therefore h(t) = \frac{f(t)}{S(t)} = \frac{\phi\left(\frac{\log t-\mu}{\sigma}\right)}{\sigma t\left\{1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right\}} \#$$

+1 (iii) Derive the equation that the turning point $t^*$ for $h(t)$ satisfies

$$\log h(t) = -\frac{(\log t-\mu)^2}{2\sigma^2} - \log(\sigma t\sqrt{2\pi}) - \log\left(1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)\right)$$

$$\frac{d\log h(t)}{dt} = \frac{-1}{\sigma^2 t}(\log t-\mu) - \frac{\cancel{\sqrt{2\pi}\sigma}}{\sqrt{2\pi}\sigma t} - \left[\frac{-\phi\left(\frac{\log t-\mu}{\sigma}\right)\cdot\frac{1}{\sigma t}}{1-\Phi\left(\frac{\log t-\mu}{\sigma}\right)}\right] = 0$$

$$\Rightarrow h(t^*) = \frac{1}{\sigma^2 t^*}(\log t^* - \mu + \sigma^2) \#$$

$h(t^*) = \frac{1}{\sigma^2 t^*}(\log t^* - \mu + \sigma^2)$

+1 (iv) Prove $e^{\mu+\sigma^2/2} \le t^*$. $e^{\mu-\sigma^2}$

$$h(t^*) = \frac{1}{\sigma^2 t^*}(\log t^* - \mu + \sigma^2) \ge 0$$

$$\Rightarrow \log t^* \ge \mu - \sigma^2$$

$$t^* \ge e^{\mu-\sigma^2}$$