Homework #3

#1

(i) In this case, assume that the brake pad life time T follows log-normal distribution with unknown parameters μ and σ. The likelihood function is $L(\mu,\sigma)=\prod_{i=1}^{n}\frac{f(t_i,\mu,\sigma)}{S(u_i,\mu,\sigma)}$ and log-likelihood function is $\ell(\mu,\sigma)=\sum_{i=1}^{n}\log(f(t_i,\mu,\sigma))-\log(S(u_i,\mu,\sigma))$, where $f(t_i,\mu,\sigma)=\frac{1}{\sqrt{2\pi}\sigma t_i}\phi(\frac{\log t_i-\mu}{\sigma})$ and $S(u_i,\mu,\sigma)=1-\Phi(\frac{\log u_i-\mu}{\sigma})$. Therefore,

$$\ell(\mu,\sigma)=-\frac{n}{2}\log(2\pi)-\frac{n}{2}\log(\sigma^2)-\sum_{i=1}^{n}\log(t_i)-\sum_{i=1}^{n}\frac{(\log(t_i)-\mu)^2}{2\sigma^2}-\log(1-\Phi(\frac{\log u_i-\mu}{\sigma})).$$

To apply the Newton-Raphson Algorithm, we should derive the score function

$$U(\mu,\sigma)=\begin{bmatrix}\frac{\partial}{\partial\mu}\ell(\mu,\sigma)\\ \frac{\partial}{\partial\sigma}\ell(\mu,\sigma)\end{bmatrix}$$ and Hessian matrix $$H(\mu,\sigma)=\begin{bmatrix}\frac{\partial^2}{\partial\mu^2}\ell(\mu,\sigma) & \frac{\partial^2}{\partial\sigma\partial\mu}\ell(\mu,\sigma)\\ \frac{\partial^2}{\partial\mu\partial\sigma}\ell(\mu,\sigma) & \frac{\partial^2}{\partial\sigma^2}\ell(\mu,\sigma)\end{bmatrix}.$$

$$\frac{\partial}{\partial\mu}\ell(\mu,\sigma)=\sum_{i=1}^{n}\frac{(\log t_i-\mu)}{\sigma^2}-\frac{1}{\sigma}\sum_{i=1}^{n}\frac{\phi(\frac{\log u_i-\mu}{\sigma})}{1-\Phi(\frac{\log u_i-\mu}{\sigma})}$$

$$\frac{\partial}{\partial\sigma}\ell(\mu,\sigma)=-\frac{n}{\sigma}+\sum_{i=1}^{n}\frac{(\log t_i-\mu)^2}{\sigma^3}-\sum_{i=1}^{n}\frac{(\frac{\log u_i-\mu}{\sigma^2})\phi(\frac{\log u_i-\mu}{\sigma})}{1-\Phi(\frac{\log u_i-\mu}{\sigma})}$$

$$\frac{\partial^2}{\partial\mu^2}\ell(\mu,\sigma)=-\frac{n}{\sigma^2}-\frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{(\log u_i-\mu)\phi(\frac{\log u_i-\mu}{\sigma})(1-\Phi(\frac{\log u_i-\mu}{\sigma})-(\phi(\frac{\log u_i-\mu}{\sigma}))^2)}{(1-\Phi(\frac{\log u_i-\mu}{\sigma}))^2}$$

$$\frac{\partial^2}{\partial\sigma^2}\ell(\mu,\sigma)=\frac{n}{\sigma^2}-\frac{3\sum_{i=1}^{n}(\log t_i-\mu)^2}{\sigma^4}+\frac{2}{\sigma^3}\sum_{i=1}^{n}\frac{(\log u_i-\mu)\phi(\frac{\log u_i-\mu}{\sigma})}{1-\Phi(\frac{\log u_i-\mu}{\sigma})}$$

$$-\sum_{i=1}^{n}\frac{(\log u_i-\mu)^2}{\sigma^4}(\frac{(\log u_i-\mu)\phi(\frac{\log u_i-\mu}{\sigma})(1-\Phi(\frac{\log u_i-\mu}{\sigma})-(\phi(\frac{\log u_i-\mu}{\sigma}))^2)}{(1-\Phi(\frac{\log u_i-\mu}{\sigma}))^2})$$

$$\frac{\partial^2}{\partial\mu\partial\sigma}\ell(\mu,\sigma)=-\frac{2}{\sigma^3}\sum_{i=1}^{n}(\log t_i-\mu)+\frac{1}{\sigma^2}\sum_{i=1}^{n}\frac{\phi(\frac{\log u_i-\mu}{\sigma})}{1-\Phi(\frac{\log u_i-\mu}{\sigma})}$$

$$-\sum_{i=1}^{n}\frac{(\log u_i-\mu)}{\sigma^3}\frac{(\frac{\log u_i-\mu}{\sigma})\phi(\frac{\log u_i-\mu}{\sigma})(1-\Phi(\frac{\log u_i-\mu}{\sigma}))-(\phi(\frac{\log u_i-\mu}{\sigma}))^2}{(1-\Phi(\frac{\log u_i-\mu}{\sigma}))^2}$$

Hence we could apply the following algorithm to obtain the MLE. The following table shows the result if the Newton-Raphson algorithm. The MLE of μ is about 4.109 and σ is about 0.421.

$\varepsilon=\lvert\theta_j-\theta_{j-1}\rvert<10^{-10}$				
STEP	μ	$\lvert\mu_j-\mu_{j-1}\rvert$	σ	$\lvert\sigma_j-\sigma_{j-1}\rvert$
0	4.00000	-	0.3000000	-
1	4.080210	8.021027e-02	0.3488822	4.888225e-02
2	4.104229	2.401824e-02	0.3942647	4.538244e-02
3	4.109126	4.897348e-03	0.4169719	2.270719e-02
4	4.109192	6.598783e-05	0.4211433	4.171405e-03
5	4.109176	1.588903e-05	0.4212609	1.176354e-04
6	4.109176	2.331113e-08	0.4212610	9.084697e-08
7	**4.109176**	**1.865175e-14**	**0.421261**	**5.484502e-14**

《Newton-Raphson Algorithm》

STEP 1. Given an initial value θ_0.

STEP 2. $\theta_j=\theta_{j-1}-H(\theta_{j-1})^{-1}\,\mathrm{U}(\theta_{j-1})$, $j=1,2,\ldots$

STEP 3. If $\theta_j-\theta_{j-1}\approx 0$, $\hat{\theta}=\theta_j$ and STOP. Otherwise, go back to STEP 2.

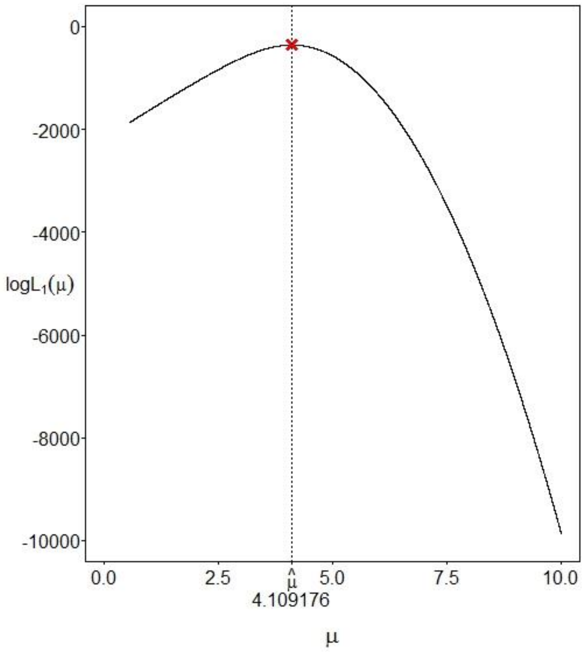
(ii) The profile likelihood of μ and σ are

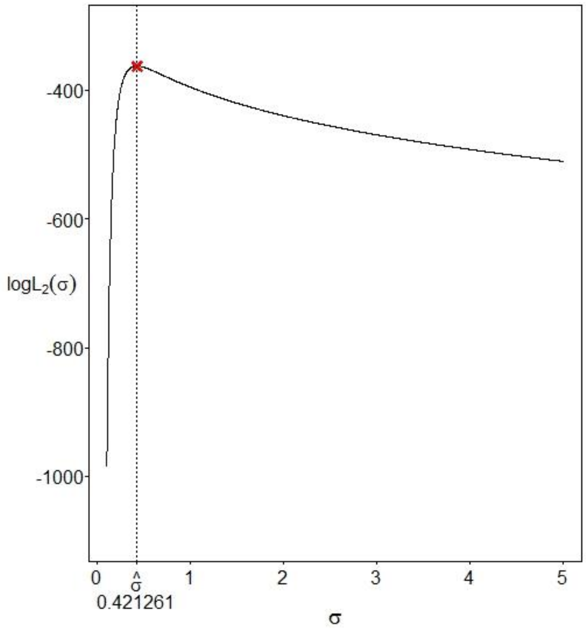
$$\ell_1(\mu)=\ell(\mu,\hat{\sigma})=-49\log(2\pi)-49\log(\hat{\sigma}^2)-\sum_{i=1}^{98}\log(t_i)-\sum_{i=1}^{98}\frac{(\log(t_i)-\mu)^2}{2\hat{\sigma}^2}-\log(1-\Phi(\frac{\log u_i-\mu}{\hat{\sigma}}))$$

$$\ell_2(\sigma)=\ell(\hat{\mu},\sigma)=-49\log(2\pi)-49\log(\sigma^2)-\sum_{i=1}^{98}\log(t_i)-\sum_{i=1}^{98}\frac{(\log(t_i)-\hat{\mu})^2}{2\sigma^2}-\log(1-\Phi(\frac{\log u_i-\hat{\mu}}{\sigma}))$$

, where $\hat{\sigma}=0.421$ and $\hat{\mu}=4.109$, respectively. Also the following plots show that

$\hat{\mu}=\max_{\mu}\ell_1(\mu)=4.10976$ and $\hat{\sigma}=\max_{\sigma}\ell_2(\sigma)=0.421261$.





《**CODE**》

```
rm(list=ls(all=TRUE))

u = c(22.2, 23.0, 24.0, 28.6, 21.8,17.0, 26.0, 23.2, 18.9, 21.9,27.3, 13.8, 24.0, 20.1, 15.7,26.8,
27.9, 15.3, 28.8, 16.0,23.6, 53.8, 21.7, 28.8, 17.0, 16.5, 15.7, 28.0, 13.3, 16.5,24.2, 17.6, 27.8,
18.3, 17.7,20.0, 13.2, 16.9, 14.9, 15.5,7.0, 15.8, 15.0, 38.3, 11.2,38.2, 26.7, 17.1, 29.0, 18.3,18.4,
18.2, 15.9, 16.4, 23.6,19.2, 23.3, 20.4, 20.9, 28.5,23.2, 17.9, 46.1, 39.3, 11.8,17.7, 30.9, 22.4,
45.0, 18.2,30.2, 21.8, 18.2, 23.0, 27.2,10.9, 25.5, 12.4, 39.9, 17.7,26.3, 14.1, 21.0, 11.2, 10.8,25.7,
32.4, 13.6, 19.1, 16.1,53.3, 57.3, 36.5, 19.7, 20.8,30.8, 20.0, 39.6)

t = c(38.7, 49.2, 42.4, 73.8, 46.7,44.1, 61.9, 39.3, 49.8, 46.3,56.2, 50.5, 54.9, 54.0, 49.2,44.8,
72.2, 107.8,81.6, 45.2,124.6,64.0, 83.0, 143.6,43.4,69.6, 74.8, 32.9, 51.5, 31.8,77.6, 63.7, 83.0,
24.8, 68.8,68.8, 89.1, 65.0, 65.1, 59.3,53.9, 79.4, 47.4, 61.4, 72.8,54.0, 37.2, 44.2, 50.8, 65.5,86.7,
43.8,100.6, 67.6, 89.5,60.3, 103.6,82.6, 88.0, 42.4,68.9, 95.7, 78.1, 83.6, 18.6,92.6, 42.4,
34.3,105.6, 20.8,52.0, 77.2, 68.9, 78.7,165.5,79.5, 55.0, 46.8,124.5, 92.5,110.0,101.2,59.4, 27.8,
33.6,69.0, 75.2, 58.4,105.6, 56.2,55.9, 83.8,123.5, 69.0,101.9,87.6, 38.8, 74.7)

lu = log(u)
lt = log(t)

###### Score Function ######
U_fun = function(theta){
	n = length(t)
```

```
	mu = theta[1]
	sigma = theta[2]
	a11 = (sum(lt-theta[1])/theta[2]^2)-(sum(dnorm((lu-mu)/sigma)/(1-pnorm((lu-mu)/sigma)))/theta[2])
	a21 = (-n/theta[2])+sum((lt-theta[1])^2/theta[2]^3)-sum(((lu-theta[1])/theta[2]^2)*dnorm((lu-mu)/sigma)/(1-pnorm((lu-mu)/sigma)))
	c(a11,a21)
}

###### Hessian Matrix ######
H_fun = function(theta){
	n = length(t)
	mu = theta[1]
	sigma = theta[2]
	a11 = -n/theta[2]^2-(1/theta[2]^2)*sum((((lu-theta[1])/theta[2])*dnorm((lu-mu)/sigma)*(1-pnorm((lu-mu)/sigma))-(dnorm((lu-mu)/sigma))^2)/(1-pnorm((lu-mu)/sigma))^2)
	a12 = (-2*sum(lt-theta[1])/theta[2]^3)+(1/theta[2]^2)*sum(dnorm((lu-mu)/sigma)/(1-pnorm((lu-mu)/sigma)))-sum(((lu-theta[1])/theta[2]^3)*(((lu-theta[1])/theta[2])*dnorm((lu-mu)/sigma)*(1-pnorm((lu-mu)/sigma))-(dnorm((lu-mu)/sigma))^2)/(1-pnorm((lu-mu)/sigma))^2)
	a22 = (n/theta[2]^2)-(3*sum((lt-theta[1])^2)/theta[2]^4)+sum((2*(lu-theta[1])/theta[2]^3)*dnorm((lu-mu)/sigma)/(1-pnorm((lu-mu)/sigma)))-sum((((lu-theta[1])/theta[2]^2)^2)*(((lu-theta[1])/theta[2])*dnorm((lu-mu)/sigma)*(1-pnorm((lu-mu)/sigma))-(dnorm((lu-mu)/sigma))^2)/(1-pnorm((lu-mu)/sigma))^2)
	matrix(c(a11,a12,a12,a22),2,2)
}

###### Newton-Raphson Algorithm ######
theta1 = c(4,0.3)
theta2 = theta1 - solve(H_fun(theta1))%*%U_fun(theta1)
mu1 = theta1[1]
mu2 = theta2[1]
sigma1 = theta1[2]
sigma2 = theta2[2]
s = 1
M = c()
S = c()
M[s] = mu1
S[s] = sigma1
while(abs(mu1-mu2)>10^(-10) || abs(sigma1-sigma2)>10^(-10)){
	theta1 = theta2
	mu1 = theta1[1]
	sigma1 = theta1[2]
	s = s + 1
	M[s] = mu1
	S[s] = sigma1
	theta2 = theta1 - (solve(H_fun(theta1)))%*%U_fun(theta1)
	mu2 = theta2[1]
	sigma2 = theta2[2]
}

###### Profile likelihood of mu ######
L1_fun = function(t1){
	n = length(t)
```

```
	sigma = sigma2
	zz = c()
	for(i in 1:length(t1)){
		zz[i] = (-n/2)*log(sigma^2)-sum(lt)-(1/(2*sigma^2))*sum((lt-t1[i])^2)-sum(log(1-
pnorm((lu-t1[i])/sigma)))
	}
	zz
}

###### Profile likelihood of sigma ######
L2_fun = function(t2){
	n = length(t)
	mu = mu2
	zz = c()
	for(i in 1:length(t2)){
		zz[i] = (-n/2)*log(t2[i]^2)-sum(lt)-(1/(2*t2[i]^2))*sum((lt-mu)^2)-sum(log(1-
pnorm((lu-mu)/t2[i])))
	}
	zz
}
###### Plot Profile likelihood of mu######
plot(seq(0,10,0.001),L1_fun(seq(0,10,0.001)),type="l",xlab="",ylab="",ylim=c(-
10000,0),axes=FALSE)
points(theta2[1],L1_fun(theta2[1]),pch=4)
axis(1,at=c(seq(0,10,2.5)),mgp=c(3,0.2,0),tcl=-0.15)
axis(1,at=theta2[1],labels=expression(hat(mu)),mgp=c(3,0.5,0),tcl=-0.15)
axis(1,at=theta2[1],mgp=c(3,1.2,0),tcl=-0.15)
axis(2,las=1,font.axis=1,at=c(seq(-10000,0,2000)),mgp=c(3,0.2,0),tcl=-0.15)
box()
mtext(side=1,line=3,expression(mu),cex=1.2)
mtext(side=2,las=1,line=0.5,expression(logL[1](mu)),cex=1)
abline(v=theta2[1],lty=3)
###### Plot Profile likelihood of sigma######
plot(seq(0.1,5,0.001),L2_fun(seq(0.1,5,0.001)),type="l",xlab="",ylab="",ylim=c(-1100,-
300),axes=FALSE)
points(theta2[2],L2_fun(theta2[2]),pch=4)
axis(1,at=c(seq(0,5,1)),mgp=c(3,0.2,0),tcl=-0.15)
axis(1,at=theta2[2],labels=expression(hat(sigma)),mgp=c(3,0.5,0),tcl=-0.15)
axis(1,at=theta2[2],mgp=c(3,1.3,0),tcl=-0.15)
axis(2,las=1,font.axis=1,at=c(seq(-1000,-400,200)),mgp=c(3,0.2,0),tcl=-0.15)
box()
mtext(side=1,line=2,expression(sigma),cex=1.2)
mtext(side=2,las=1,line=0.5,expression(logL[2](sigma)),cex=1)
abline(v=theta2[2],lty=3)
```

《OUTPUT》

```
> mu2
[1] 4.109176
> sigma2
[1] 0.421261
```