HW#8 Survival Analysis II

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<u>Exercise 14.8 (p.251)</u>

1.

Consider Freund's bivariate exponential distribution

$$\bar{G}(t_1, t_2) = \begin{cases} \dfrac{1}{\lambda_+ - \mu_2}\{\lambda_1 e^{-(\lambda_+ - \mu_2)t_1 - \mu_2 t_2} + (\lambda_2 - \mu_2)e^{-\lambda_+ t_2}\} & \text{if } t_1 \le t_2, \\ \dfrac{1}{\lambda_+ - \mu_1}\{\lambda_2 e^{-(\lambda_+ - \mu_1)t_2 - \mu_1 t_1} + (\lambda_1 - \mu_1)e^{-\lambda_+ t_1}\} & \text{if } t_1 \ge t_2, \end{cases}$$

where $\lambda_+ = \lambda_1 + \lambda_2$, $t_1 > 0$ and $t_2 > 0$.

If the risks independent, we have

$$h(j, t) = h_j(t), \quad \text{for} \quad j = 1, 2.$$

Therefore, we need to check under what parametric conditions satisfy $h(j, t) = h_j(t)$, for $j = 1, 2$. Consider $j = 1$, by Tsiatis's Lemma we have

$$\begin{aligned} f(1, t) &= -\left.\frac{\partial \bar{G}(\mathbf{t})}{\partial t_1}\right|_{\mathbf{t} = t\mathbf{1}_p} \\ &= \left.\frac{-1}{(\lambda_+ - \mu_2)}\{\lambda_1 e^{-(\lambda_+ - \mu_2)t_1 - \mu_2 t_2}(\lambda_+ - \mu_2)\}\right|_{\mathbf{t} = t\mathbf{1}_p} \\ &= \lambda_1 e^{-\lambda_+ t_1}. \end{aligned}$$

Then

$$\begin{aligned} h(1, t) &= \frac{f(1, t)}{\bar{F}(t)} \\ &= \frac{\lambda_1 e^{-\lambda_+ t_1}}{\dfrac{1}{(\lambda_+ - \mu_2)}\{\lambda_1 e^{-\lambda_+ t_1} + (\lambda_2 - \mu_2)e^{-\lambda_+ t_1}\}} \\ &= \frac{\lambda_1(\lambda_+ - \mu_2)}{\lambda_1 + \lambda_2 - \mu_2} \\ &= \lambda_1. \end{aligned}$$

Hence we have obtained $h(1, t) = \lambda_1$.

Now we compute $h_1(t) = \frac{d}{dt}\{-\log \overline{G}_1(t)\}$.

$$\begin{aligned}\overline{G}_1(t_1) &= \overline{G}(t_1, 0) \\ &= \frac{1}{\lambda_+ - \mu_1}\{\lambda_2 e^{-\mu_1 t_1} + (\lambda_1 - \mu_1)e^{-\lambda_+ t_1}\}.\end{aligned}$$

Then

$$\frac{d}{dt}\{-\log \overline{G}_1(t)\} = \frac{\lambda_2 \mu_1 e^{-\mu_1 t} + (\lambda_1 - \mu_1)\lambda_+ e^{-\lambda_+ t}}{\lambda_2 e^{-\mu_1 t} + (\lambda_1 - \mu_1)e^{-\lambda_+ t}}.$$

Hence we have obtained

$$h_1(t) = \frac{\lambda_2 \mu_1 e^{-\mu_1 t} + (\lambda_1 - \mu_1)\lambda_+ e^{-\lambda_+ t}}{\lambda_2 e^{-\mu_1 t} + (\lambda_1 - \mu_1)e^{-\lambda_+ t}}.$$

Therefore, we set

$$\begin{aligned}&h(1,t) = h_1(t) \\ &\Rightarrow \lambda_1 = \frac{\lambda_2 \mu_1 e^{-\mu_1 t} + (\lambda_1 - \mu_1)\lambda_+ e^{-\lambda_+ t}}{\lambda_2 e^{-\mu_1 t} + (\lambda_1 - \mu_1)e^{-\lambda_+ t}} \\ &\Rightarrow \lambda_1 \lambda_2 e^{-\mu_1 t} + (\lambda_1 - \mu_1)\lambda_1 e^{-\lambda_+ t} = \lambda_2 \mu_1 e^{-\mu_1 t} + (\lambda_1 - \mu_1)\lambda_+ e^{-\lambda_+ t} \\ &\Rightarrow (\lambda_1 - \mu_1)\lambda_2 e^{-\mu_1 t} = (\lambda_1 - \mu_1)(\lambda_+ - \lambda_1)e^{-\lambda_+ t} \\ &\Rightarrow (\lambda_1 - \mu_1)e^{-\mu_1 t} = (\lambda_1 - \mu_1)e^{-\lambda_+ t} \\ &\Rightarrow (\lambda_1 - \mu_1)e^{-\mu_1} = (\lambda_1 - \mu_1)e^{-\lambda_+}.\end{aligned}$$

Hence if $\lambda_1 = \mu_1$, we have $h(1,t) = h_1(t)$. Similarly, for $j = 2$. The risks independent if

$$\lambda_2 = \mu_2.$$

Thus, we have proven that under the conditions $\lambda_j = \mu_j$, for $j = 1, 2$ the risks independent.

2. (i)

Consider Multivariate Burr (MB) distribution

$$\overline{G}(\mathbf{t}) = (1+s)^{-\upsilon},$$

where $s = \sum_{i=1}^{p} (t_i / \xi_i)^{\phi_i}$, and $\upsilon > 0$ is the shape parameter of a gamma frailty.

If the risks independent, we have

$$h(j, t) = h_j(t), \quad \text{for} \quad j = 1, \cdots, p.$$

Therefore, we need to check under what parametric conditions satisfy

$h(j, t) = h_j(t)$, for $j = 1, \cdots, p$. Consider $j = 1$, by Tsiatis's Lemma we have

$$f(1, t) = \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} (1+s_t)^{-\upsilon-1},$$

where $s_t = \sum_{i=1}^{p} (t / \xi_i)^{\phi_i}$.

Then

$$\begin{aligned} h(1, t) &= \frac{f(1, t)}{\overline{F}(t)} \\ &= \frac{\frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} (1+s_t)^{-\upsilon-1}}{(1+s_t)^{-\upsilon}} \\ &= \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} (1+s_t)^{-1}. \end{aligned}$$

Hence we have obtained

$$h(1, t) = \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} (1+s_t)^{-1}.$$

Now we compute $h_1(t) = \frac{d}{dt}\{-\log \overline{G}_1(t)\}$.

$$\overline{G}_1(t_1) = \overline{G}(t_1, 0, \cdots, 0)$$
$$= \left\{1 + \left(\frac{t_1}{\xi_1}\right)^{\phi_1}\right\}^{-\upsilon}.$$

Then

$$-\log \overline{G}_1(t) = \upsilon \log\left\{1 + \left(\frac{t}{\xi_1}\right)^{\phi_1}\right\}$$

and

$$\frac{d}{dt}\{-\log \overline{G}_1(t)\} = \frac{\upsilon\phi_1(t/\xi_1)^{\phi_1 - 1}}{\xi_1\{1 + (t/\xi_1)^{\phi_1}\}}$$
$$= \frac{\upsilon\phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1}\{1 + (t/\xi_1)^{\phi_1}\}^{-1}.$$

Hence we have obtained

$$h_1(t) = \frac{\upsilon\phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1}\{1 + (t/\xi_1)^{\phi_1}\}^{-1}.$$

Therefore, we set

$$h(1, t) = h_1(t)$$
$$\Rightarrow \frac{\upsilon\phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1}\left\{1 + \sum_{i=1}^{p}(t/\xi_i)^{\phi_i}\right\}^{-1} = \frac{\upsilon\phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1}\{1 + (t/\xi_1)^{\phi_1}\}^{-1}$$
$$\Rightarrow 1 = \frac{1 + \sum_{i=1}^{p}(t/\xi_i)^{\phi_i}}{1 + (t/\xi_1)^{\phi_1}}$$
$$\Rightarrow 1 + \sum_{i=1}^{p}(t/\xi_i)^{\phi_i} = 1 + (t/\xi_1)^{\phi_1}$$
$$\Rightarrow \sum_{i=1}^{p}(t/\xi_i)^{\phi_i} = (t/\xi_1)^{\phi_1}$$
$$\Rightarrow \sum_{i \neq 1}(t/\xi_i)^{\phi_i} = 0.$$

Hence the condition dose not hold as long as $\upsilon > 0$.

Consider the case

$$\overline{G}(t_1, t_2) = \overline{G}(t_1)\overline{G}(t_2)$$

$$\Rightarrow \left\{1+\left(\frac{t_1}{\xi_1}\right)^{\phi_1}+\left(\frac{t_2}{\xi_2}\right)^{\phi_2}\right\}^{-\upsilon} = \left\{1+\left(\frac{t_1}{\xi_1}\right)^{\phi_1}\right\}^{-\upsilon}\left\{1+\left(\frac{t_2}{\xi_2}\right)^{\phi_2}\right\}^{-\upsilon}$$

$$\Rightarrow \left\{1+\left(\frac{t_1}{\xi_1}\right)^{\phi_1}+\left(\frac{t_2}{\xi_2}\right)^{\phi_2}\right\}^{-\upsilon} = \left\{1+\left(\frac{t_1}{\xi_1}\right)^{\phi_1}+\left(\frac{t_2}{\xi_2}\right)^{\phi_2}+\left(\frac{t_1}{\xi_1}\right)^{\phi_1}\left(\frac{t_2}{\xi_2}\right)^{\phi_2}\right\}^{-\upsilon}.$$

The equality is satisfied if $\left(\frac{t_1}{\xi_1}\right)^{\phi_1}\left(\frac{t_2}{\xi_2}\right)^{\phi_2} = 0$ or $\upsilon \to 0$. Since $\left(\frac{t_1}{\xi_1}\right)^{\phi_1}\left(\frac{t_2}{\xi_2}\right)^{\phi_2}$ can not be 0. Therefore, the risk independent if $\upsilon \to 0$.

2. (ii)

Consider Multivariate Weibull (MW) distribution

$$\overline{G}(\mathbf{t}) = \exp(-s^{\upsilon}),$$

where $s = \sum_{i=1}^{p} (t_i / \xi_i)^{\phi_i}$.

If the risks independent, we have

$$h(j, t) = h_j(t), \quad \text{for} \quad j = 1, \cdots, p.$$

Therefore, we need to check under what parametric conditions satisfy $h(j, t) = h_j(t)$, for $j = 1, \cdots, p$. Consider $j = 1$, by Tsiatis's Lemma we have

$$f(1, t) = \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} s_t^{\upsilon - 1} \exp(-s_t^{\upsilon}),$$

where $s_t = \sum_{i=1}^{p} (t / \xi_i)^{\phi_i}$.

Then

$$\begin{aligned} h(1, t) &= \frac{f(1, t)}{\overline{F}(t)} \\ &= \frac{\frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} s_t^{\upsilon - 1} \exp(-s_t^{\upsilon})}{\exp(-s_t^{\upsilon})} \\ &= \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} s_t^{\upsilon - 1}. \end{aligned}$$

Hence we have obtained

$$h(1, t) = \frac{\upsilon \phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1} s_t^{\upsilon - 1}.$$

Now we compute $h_1(t) = \frac{d}{dt}\{-\log \overline{G}_1(t)\}$.

$$\begin{aligned} \overline{G}_1(t_1) &= \overline{G}(t_1, 0, \cdots, 0) \\ &= \exp\left\{-\left(\frac{t_1}{\xi_1}\right)^{\upsilon\phi_1}\right\}. \end{aligned}$$

Then

$$-\log \overline{G}_1(t) = \left(\frac{t}{\xi_1}\right)^{\upsilon\phi_1}$$

and

$$\frac{d}{dt}\{-\log \overline{G}_1(t)\} = \frac{\upsilon\phi_1}{\xi_1}\left(\frac{t_1}{\xi_1}\right)^{\upsilon\phi_1 - 1}.$$

Hence we have obtained

$$h_1(t) = \frac{\upsilon\phi_1}{\xi_1}\left(\frac{t_1}{\xi_1}\right)^{\upsilon\phi_1 - 1}.$$

Therefore, we set

$$\begin{aligned} & h(1, t) = h_1(t) \\ \Rightarrow\ & \frac{\upsilon\phi_1}{\xi_1^{\phi_1}} t^{\phi_1 - 1}\left\{\sum_{i=1}^{p}(t/\xi_i)^{\phi_i}\right\}^{\upsilon - 1} = \frac{\upsilon\phi_1}{\xi_1}\left(\frac{t_1}{\xi_1}\right)^{\upsilon\phi_1 - 1} \\ \Rightarrow\ & \frac{1}{\xi_1^{\phi_1 - 1}} = \frac{(t_1/\xi_1)^{\upsilon\phi_1 - 1}}{\left\{\sum_{i=1}^{p}(t/\xi_i)^{\phi_i}\right\}^{\upsilon - 1}} \\ \Rightarrow\ & \xi_1^{\phi_1 - 1} = \frac{\left\{\sum_{i=1}^{p}(t/\xi_i)^{\phi_i}\right\}^{\upsilon - 1}}{(t_1/\xi_1)^{\upsilon\phi_1 - 1}}. \end{aligned}$$

Let $\upsilon = 1$, we have

$$\begin{aligned}
&\xi_1^{\phi_1 - 1} = \frac{1}{(t_1 / \xi_1)^{\phi_1 - 1}} \\
&\Rightarrow \xi_1^{\phi_1 - 1} = \frac{\xi_1^{\phi_1 - 1}}{t_1^{\phi_1 - 1}} \\
&\Rightarrow t_1^{\phi_1 - 1} = 1.
\end{aligned}$$

Similarly, for $j = 1, \cdots, p$. The risks independent if $\upsilon = 1$ and $\phi_j = 1$

Thus, we have proven that under the conditions $\upsilon = 1$ and $\phi_j = 1$ for $j = 1, \cdots, p$ the risks independent.