HW#2 Survival analysis II, Spring 2015, Due 3/10 (Tue)
Name:

2.7.1 Exercise (p.22)

- Answer 2 + Include the formula of $E(T^r)$
- Answer 3 + Include the formulas of survivor and hazard functions
- Answer 7
- Answer 8
- Answer 9 + Include a different way from the solution (use Fubini's thoerem)

2.7.2. Exercise (p.23)

- Answer 1. + Answer for the case of series system (p.20).

Name: Pan, Chi-Hung 102225011
2.7.1

2. Let T is the Weibull distribution with distribution and density function are

$$F(t) = 1 - \exp\{-(t/\xi)^{\nu}\}$$

$$f(t) = \frac{\nu}{\xi^{\nu}} t^{\nu-1} \exp\{-(t/\xi)^{\nu}\}.$$

The formula of $E(T^r)$ is

$$E(T^r) = \int_0^{\infty} t^r \frac{\nu}{\xi^{\nu}} t^{\nu-1} \exp\{-(t/\xi)^{\nu}\} dt = \frac{\nu}{\xi^{\nu}} \int_0^{\infty} t^{r+\nu-1} \exp(-t^{\nu}/\xi^{\nu}) dt$$

Let $t^{\nu} = x, \nu t^{\nu-1} dt = dx$

$$= \frac{\nu}{\xi^{\nu}} \int_0^{\infty} x^{(r+\nu-1)/\nu} \exp(-x/\xi^{\nu}) \frac{x^{(1-\nu)/\nu}}{\nu} dx$$

$$= \frac{1}{\xi^{\nu}} \int_0^{\infty} x^{r/\nu} \exp(-x/\xi^{\nu}) dx$$

$$= \frac{\Gamma(r/\nu+1)(\xi^{\nu})^{r/\nu+1}}{\xi^{\nu}} \int_0^{\infty} \frac{1}{\Gamma(r/\nu+1)(\xi^{\nu})^{r/\nu+1}} x^{r/\nu+1-1} \exp(-x/\xi^{\nu}) dx$$

$$= \Gamma(r/\nu+1)\xi^{r} \times 1$$

The mean of T is

$$E(T) = \Gamma(1/\nu+1)\xi.$$

The variance of T is

$$Var(T) = E(T^2) - \{E(T)\}^2$$
$$= \Gamma(2/\nu+1)\xi^2 - \{\Gamma(1/\nu+1)\xi\}^2.$$

The upper q-th quantile is

$$\Pr(T > t_q) = F(t_q) = 1 - \exp\{-(t_q/\xi)^{\nu}\} = q$$

$$\Rightarrow \exp\{-(t_q/\xi)^{\nu}\} = 1 - q$$
$$\Rightarrow -(t_q/\xi)^{\nu} = \log(1-q)$$
$$\Rightarrow t_q = \xi\{-\log(1-q)\}^{\frac{1}{\nu}}$$

3.Let $Y \sim N(\mu, \sigma^2)$ and $T = e^Y$, then distribution of T is

$$\Pr(T \le t) = \Pr(\log T \le \log t) = \Pr(Y \le \log t) = \Pr\left(\frac{Y-\mu}{\sigma} \le \frac{\log t - \mu}{\sigma}\right) = \Phi\left(\frac{\log t - \mu}{\sigma}\right),$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\} dx$.

The formula of survival function is:

$$\bar{F}(t) = 1 - \Phi\left(\frac{\log t - \mu}{\sigma}\right).$$

The density of

$$f(t) = \frac{d\Phi\left(\frac{\log t - \mu}{\sigma}\right)}{dt} = \frac{1}{t\sigma}\phi\left(\frac{\log t - \mu}{\sigma}\right),$$

where $\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{x^2}{2}\right\}$.

The formula of hazard function is:

$$h(t) = \frac{f(t)}{\bar{F}(t)} = \frac{\frac{1}{t\sigma}\phi\left(\frac{\log t - \mu}{\sigma}\right)}{1 - \Phi\left(\frac{\log t - \mu}{\sigma}\right)}.$$

The formula of $E(T^r)$ is

$$E(T^r) = E(e^{Yr}) \quad \text{by moment generating function of Normal distribution}$$

$$= \exp\left(\mu r + \frac{1}{2}\sigma^2 r^2\right)$$

The mean of T is

$$E(T) = \exp\left(\mu + \frac{1}{2}\sigma^2\right).$$

The variance of T is

$$\begin{aligned} Var(T) &= E(T^2) - \{E(T)\}^2 \\ &= \exp(2\mu + 2\sigma^2) - \exp(2\mu + \sigma^2). \end{aligned}$$

7.(a) Prove: If $\int_0^\infty h(u)du < \infty$, then $\Pr(T > \infty.) > 0$.

$$\Pr(T > t) = \bar{F}(t) = \exp\left(-\int_0^t h(u)du\right)$$

$$\because \int_0^t h(u)du < \infty \text{ as } t \to \infty$$

$$\therefore \Pr(T > t) > 0 \text{ as } t \to \infty.$$

(b)Prove: If $h(t) = h_1(t) + h_2(t)$, where $h_1(t)$ and $h_2(t)$ are the hazard functions of independent failure time T_1 and T_2, then T has the same distribution as $\min(T_1, T_2)$.

$$\Pr(T > t) = \bar{F}(t) = \exp\left(-\int_0^t h(u)du\right) = \exp\left(-\int_0^t h_1(u) + h_2(u)du\right)$$

$$= \exp\left(-\int_0^t h_1(u)du\right)\exp\left(-\int_0^t h_2(u)du\right)$$

$$= \Pr(T_1 > t)\Pr(T_2 > t) \qquad (\because T_1 \perp T_2)$$

$$= \Pr(T_1 > t, T_2 > t)$$

$$= \Pr\{\min(T_1, T_2) > t\}$$

⇨ T has the same distribution as $\min(T_1, T_2)$.

8. Let T has hazard function $h(t) = a$ for $0 < t < t_0$, $h(t) = b$ for $t \ge t_0$.

$$\Pr(T > t) = \bar{F}(t) = \begin{cases} \exp\left(-\int_0^t adu\right) & \text{if } 0 < t < t_0 \\ \exp\left(-\int_0^{t_0} adu - \int_{t_0}^t bdu\right) & \text{if } t \ge t_0 \end{cases}$$

$$= \begin{cases} \exp(-at) & \text{if } 0 < t < t_0 \\ \exp\{-at_0 - b(t - t_0)\} & \text{if } t \ge t_0 \end{cases}$$

⇨ $\Pr(T > t_0) = \exp(-at_0), \Pr(T > 2t_0) = \exp\{-at_0 - b(2t_0 - t_0)\} = \exp\{-t_0(a + b)\}$

9. Show that, for continuous T, $E(T) = \int_0^\infty \bar{F}(t)dt$, provided that $t\bar{F}(t) \to 0$ *as* $t \to \infty$.

By definition: $E(T) = \int_0^\infty tf(t)dt$, where $f(t)$ is the density of T.

$$E(T)=\int_0^\infty tf(t)dt=-\int_0^\infty td\overline{F}(t) \quad \text{(integration by parts)}$$

$$=-t\overline{F}(t)\Big|_0^\infty+\int_0^\infty \overline{F}(t)dt$$

$$=\int_0^\infty \overline{F}(t)dt\,.$$

$$E(T)=\int_0^\infty tf(t)dt=\int_0^\infty\left(\int_{u=0}^{u=t}1du\right)f(t)dt=\int_0^\infty\int_0^t f(t)dudt \quad \text{(by fubini's theorem)}$$

$$=\int_0^\infty\int_u^\infty f(t)dtdu=\int_0^\infty \overline{F}(u)du$$

For discrete T ,let $\Pr(T=j)=p_j,\ j=1,2,...,\ \Pr(T>j)=q_j$; show that $E(T)=\sum_{j=0}^\infty q_j$.

$$E(T)=\sum_{j=0}^\infty j\times p_j=p_1+2p_2+3p_3+4p_4...$$

$$=p_1+p_2+p_3+p_4+...$$

$$+p_2+p_3+p_4+...$$

$$+p_3+p_4+....$$

$$=q_1+q_2+q_3+.....=\sum_{j=0}^\infty q_j$$

2.7.2

1. Let $\Pr(\text{ the j component functions })=p_j,\ j=1,2,...,n.$

For the parallel system:

$\Pr(\text{ the system fumctions })=1-\Pr(\text{ the system does not fumction })$

$=1-\Pr(\text{ all the component do not function })$

$=1-\prod_{j=1}^n(1-p_j)$.

For the series system:

$\Pr(\text{ the system fumctions })=\Pr(\text{ all the component function })=\prod_{j=1}^n p_j$.