Homework#4 Statistical Inference III

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Problem 4.3 [p.139]

Let X have the binomial distribution $b(p, n)$, and consider the hypothesis $H: p = p_0$ at level of significance α. Determine the boundary values of the UMP unbiased test for $n = 10$ with $\alpha = 0.1$, $p_0 = 0.2$ and with $\alpha = 0.05$, $p_0 = 0.4$ and in each case graph the power functions of the unbiased test.

(i) Case 1: $\alpha = 0.1$ and $p_0 = 0.2$.

Solution:

Since X follows the binomial distribution, we have

$$f_p(x) = \binom{n}{x} p^x (1-p)^{n-x} = (1-p)^n \exp\left\{ x \log\left(\frac{p}{1-p} \right) \right\} \binom{n}{x}.$$

Hence the binomial distribution is a one-parameter exponential family with $T(x) = x$. Consider the test

$$\phi(x) = \begin{cases} 1 & \text{if } x < c_1 \text{ or } x > c_2, \\ \gamma_1 & \text{if } x = c_1, \\ \gamma_2 & \text{if } x = c_2, \\ 0 & \text{if } c_1 < x < c_2, \end{cases}$$

where

$$E_{p_0}\{\phi(X)\} = \alpha \tag{1}$$

and

$$E_{p_0}\{X\phi(X)\} = \alpha E_{p_0}(X). \tag{2}$$

Then $\phi(x)$ is an UMP unbiased test. We aim to solve γ_i, c_i, $i = 1, 2$. First consider Equation (1), it is equivalent to

$$E_{p_0}\{\phi(X)\} = \alpha \Leftrightarrow E_{p_0}\{1-\phi(X)\} = 1-\alpha,$$

where

$$1-\phi(x) = \begin{cases} 0 & \text{if } x < c_1 \text{ or } x > c_2, \\ 1-\gamma_1 & \text{if } x = c_1, \\ 1-\gamma_2 & \text{if } x = c_2, \\ 1 & \text{if } c_1 < x < c_2. \end{cases}$$

Then it follows

$$\begin{aligned}
&E_{p_0}\{1-\phi(X)\} = 1-\alpha \\
&\Leftrightarrow \Pr(c_1 < X < c_2) + \Pr(X = c_1) + \Pr(X = c_2) = 1-\alpha \\
&\Leftrightarrow \sum_{x=c_1+1}^{c_2-1} \binom{n}{x} p_0^x (1-p_0)^{n-x} + \sum_{i=1}^{2} (1-\gamma_i) \binom{n}{c_i} p_0^{c_i} (1-p_0)^{n-c_i} = 1-\alpha \\
&\Leftrightarrow \sum_{x=c_1}^{c_2} \binom{n}{x} p_0^x (1-p_0)^{n-x} - \gamma_1 \binom{n}{c_1} p_0^{c_1} (1-p_0)^{n-c_1} - \gamma_2 \binom{n}{c_2} p_0^{c_2} (1-p_0)^{n-c_2} = 1-\alpha \\
&\Leftrightarrow \gamma_1 \binom{n}{c_1} p_0^{c_1} (1-p_0)^{n-c_1} + \gamma_2 \binom{n}{c_2} p_0^{c_2} (1-p_0)^{n-c_2} = \sum_{x=c_1}^{c_2} \binom{n}{x} p_0^x (1-p_0)^{n-x} - 1 + \alpha.
\end{aligned}$$

Similarly, Equation (2) is equivalent to

$$\begin{aligned}
&E_{p_0}\{X\phi(X)\} = \alpha E_{p_0}(X) \\
&\Leftrightarrow E_{p_0}[X\{1-\phi(X)\}] = (1-\alpha)E_{p_0}(X) \\
&\Leftrightarrow \sum_{x=c_1+1}^{c_2-1} x \binom{n}{x} p_0^x (1-p_0)^{n-x} + \sum_{i=1}^{2} (1-\gamma_i) c_i \binom{n}{c_i} p_0^{c_i} (1-p_0)^{n-c_i} = (1-\alpha) n p_0 \\
&\Leftrightarrow \sum_{x=c_1+1}^{c_2-1} \binom{n-1}{x-1} p_0^{x-1} (1-p_0)^{(n-1)-(x-1)} + \sum_{i=1}^{2} (1-\gamma_i) \binom{n-1}{c_i-1} p_0^{c_i-1} (1-p_0)^{(n-1)-(c_i-1)} = 1-\alpha \\
&\Leftrightarrow \sum_{x=c_1}^{c_2} \binom{n-1}{x-1} p_0^{x-1} (1-p_0)^{(n-1)-(x-1)} - \gamma_1 \binom{n-1}{c_1-1} p_0^{c_1-1} (1-p_0)^{(n-1)-(c_1-1)} \\
&\qquad\qquad - \gamma_2 \binom{n-1}{c_2-1} p_0^{c_2-1} (1-p_0)^{(n-1)-(c_2-1)} = 1-\alpha \\
&\Leftrightarrow \gamma_1 \binom{n-1}{c_1-1} p_0^{c_1-1} (1-p_0)^{(n-1)-(c_1-1)} + \gamma_2 \binom{n-1}{c_2-1} p_0^{c_2-1} (1-p_0)^{(n-1)-(c_2-1)} \\
&\qquad\qquad = \sum_{x=c_1}^{c_2} \binom{n-1}{x-1} p_0^{x-1} (1-p_0)^{(n-1)-(x-1)} - 1 + \alpha.
\end{aligned}$$

Therefore, we have shown that Equation (1) and (2) is equivalent to the functions

$$\gamma_1\binom{n}{c_1}p_0^{c_1}(1-p_0)^{n-c_1}+\gamma_2\binom{n}{c_2}p_0^{c_2}(1-p_0)^{n-c_2}=\sum_{x=c_1}^{c_2}\binom{n}{x}p_0^{x}(1-p_0)^{n-x}-1+\alpha$$

and

$$\gamma_1\binom{n-1}{c_1-1}p_0^{c_1-1}(1-p_0)^{(n-1)-(c_1-1)}+\gamma_2\binom{n-1}{c_2-1}p_0^{c_2-1}(1-p_0)^{(n-1)-(c_2-1)}$$
$$=\sum_{x=c_1}^{c_2}\binom{n-1}{x-1}p_0^{x-1}(1-p_0)^{(n-1)-(x-1)}-1+\alpha.$$

Then we can use R program to solve $\gamma_i, c_i,\ i=1,2$. Since $E_{p_0}(X)=2$, we can guess the acceptance region is around 2. The result is

$$\phi(x)=\begin{cases}1 & \text{if } x=5 \text{ or } 6 \text{ or } \cdots \text{ or } 10,\\ 0.5590 & \text{if } x=0,\\ 0.0815 & \text{if } x=4,\\ 0 & \text{if } x=1 \text{ or } 2 \text{ or } 3.\end{cases}$$

Hence we have derived an UMP unbiased test $\phi(x)$. Figure 1 plots the power function $\beta(p)=E_p\{\phi(X)\}$. It shows that the power function reaches a minimum $(\beta(p)\ge\alpha=0.1)$ at $p=p_0=0.2$.

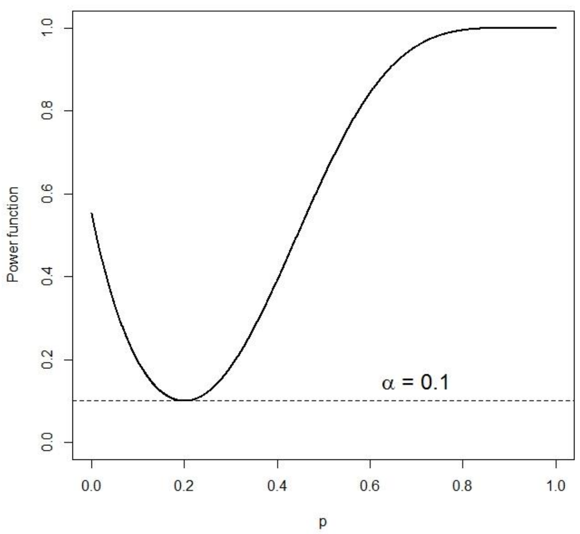

Figure 1. The power function $\beta(p)=E_p\{\phi(X)\}$, $\alpha=0.1$ and $p_0=0.2$.

(ii) Case 2: $\alpha = 0.05$ and $p_0 = 0.4$.

Solution:

In a similar fashion. Consider the test

$$\phi(x) = \begin{cases} 1 & \text{if } x < c_1 \text{ or } x > c_2, \\ \gamma_1 & \text{if } x = c_1, \\ \gamma_2 & \text{if } x = c_2, \\ 0 & \text{if } c_1 < x < c_2, \end{cases}$$

where

$$\gamma_1 \binom{n}{c_1} p_0^{c_1} (1 - p_0)^{n - c_1} + \gamma_2 \binom{n}{c_2} p_0^{c_2} (1 - p_0)^{n - c_2} = \sum_{x = c_1}^{c_2} \binom{n}{x} p_0^{x} (1 - p_0)^{n - x} - 1 + \alpha$$

and

$$\gamma_1 \binom{n-1}{c_1 - 1} p_0^{c_1 - 1} (1 - p_0)^{(n-1)-(c_1-1)} + \gamma_2 \binom{n-1}{c_2 - 1} p_0^{c_2 - 1} (1 - p_0)^{(n-1)-(c_2-1)}$$
$$= \sum_{x = c_1}^{c_2} \binom{n-1}{x-1} p_0^{x-1} (1 - p_0)^{(n-1)-(x-1)} - 1 + \alpha.$$

Again, we can use R program to solve γ_i, c_i, $i = 1, 2$. Since $E_{p_0}(X) = 4$, we can guess the acceptance region is around 4. The result is

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0 \text{ or } 8 \text{ or } 9 \text{ or } 10, \\ 0.5034 & \text{if } x = 1, \\ 0.2677 & \text{if } x = 7, \\ 0 & \text{if } x = 2 \text{ or } 3 \text{ or } 4 \text{ or } 5 \text{ or } 6. \end{cases}$$

Hence we have derived an UMP unbiased test $\phi(x)$. Figure 2 plots the power function $\beta(p) = E_p\{\phi(X)\}$. It shows that the power function reaches a minimum $(\beta(p) \ge \alpha = 0.05)$ at $p = p_0 = 0.4$.

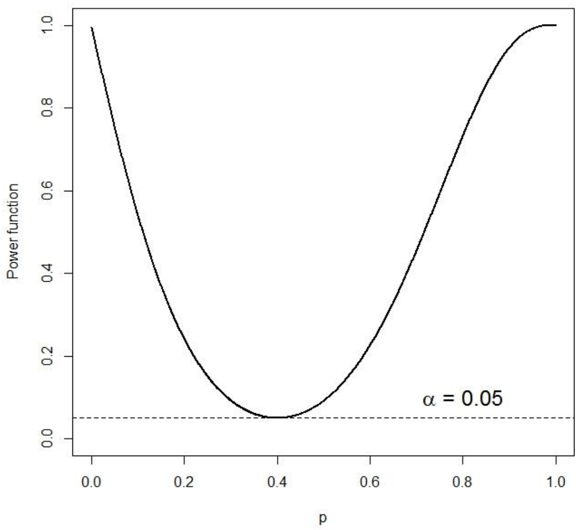

Figure 2. The power function $\beta(p) = E_p\{\phi(X)\}$, $\alpha = 0.05$ and $p_0 = 0.4$.

Problem 4.4 [p.139]

Let X have the Poisson distribution $P(\tau)$, and consider the hypothesis $H:\tau=\tau_0$. Then the condition (4.6) reduces to

$$\sum_{x=c_1+1}^{c_2-1}\frac{\tau_0^{x-1}}{(x-1)!}e^{-\tau_0}+\sum_{i=1}^{2}(1-\gamma_i)\frac{\tau_0^{c_i-1}}{(c_i-1)!}e^{-\tau_0}=1-\alpha,$$

provided $c_1>1$.

Solution:

The derivation is similar as Problem 4.3. Since X follows the Poisson distribution, we have

$$f_\tau(x)=\frac{\tau_0^x}{x!}e^{-\tau_0}=e^{-\tau_0}\exp\{x\log\tau_0\}\frac{1}{x!}.$$

Hence the Poisson distribution is a one-parameter exponential family with $T(x)=x$.

Consider the test

$$\phi(x)=\begin{cases}1 & \text{if } x<c_1 \text{ or } x>c_2,\\ \gamma_1 & \text{if } x=c_1,\\ \gamma_2 & \text{if } x=c_2,\\ 0 & \text{if } c_1<x<c_2,\end{cases}$$

where $E_{p_0}\{\phi(X)\}=\alpha$ and $E_{p_0}\{X\phi(X)\}=\alpha E_{p_0}(X)$. Consider

$$1-\phi(x)=\begin{cases}0 & \text{if } x<c_1 \text{ or } x>c_2,\\ 1-\gamma_1 & \text{if } x=c_1,\\ 1-\gamma_2 & \text{if } x=c_2,\\ 1 & \text{if } c_1<x<c_2.\end{cases}$$

Then the condition (4.6), that is, $E_{p_0}\{X\phi(X)\}=\alpha E_{p_0}(X)$ follows

$$
\begin{aligned}
&E_{p_0}\{X\phi(X)\} = \alpha E_{p_0}(X) \\
&\Leftrightarrow E_{p_0}[X\{1-\phi(X)\}] = (1-\alpha)E_{p_0}(X) \\
&\Leftrightarrow \sum_{x=c_1+1}^{c_2-1} x\frac{\tau_0^x}{x!}e^{-\tau_0} + \sum_{i=1}^{2}(1-\gamma_i)c_i\frac{\tau_0^{c_i}}{c_i!}e^{-\tau_0} = (1-\alpha)\tau_0 \\
&\Leftrightarrow \sum_{x=c_1+1}^{c_2-1} \frac{\tau_0^{x-1}}{(x-1)!}e^{-\tau_0} + \sum_{i=1}^{2}(1-\gamma_i)\frac{\tau_0^{c_i-1}}{(c_i-1)!}e^{-\tau_0} = 1-\alpha.
\end{aligned}
$$

Hence we have proven the desired results.

R codes – Figure 1

```
n        = 10
alpha = 0.1
p0      = 0.2

c1 = 0
c2 = 4

x = seq(c1,c2)
k = length(x);k
a = dbinom(x,n,p0); a; sum(a)
b = dbinom(x-1,n-1,p0); b; sum(b)

a[1]; a[k]
b[1]; b[k]
g1 = ((1-alpha)*(b[k]-a[k])-(b[k]*sum(a)-a[k]*sum(b)))/(b[1]*a[k]-a[1]*b[k]); g1
g2 = (sum(a)-g1*a[1]-(1-alpha))/a[k]; g2

sum(a)-g1*a[1]-g2*a[k]
sum(b)-g1*b[1]-g2*b[k]

Power = c()
p_v = seq(0.001,1,0.001)
i = 1
for (p in p_v) {

     E1 = sum(dbinom(c((c2+1):10),n,p))
     E2 = g1*dbinom(c1,n,p)
     E3 = g2*dbinom(c2,n,p)
     Power[i] = E1+E2+E3
     i = i+1

}

plot(p_v,Power,ylim=c(0,1),xlim=c(0,1),type="l",xlab="p",ylab="Power
function",lwd=2)
abline(h=alpha,lty=2)
text(0.7,0.15,expression(alpha*" = 0.1"),cex=1.5)
```

R codes – Figure 2

```
n        = 10
alpha = 0.05
p0       = 0.4

c1 = 1
c2 = 7

x = seq(c1,c2)
k = length(x);k
a = dbinom(x,n,p0); a; sum(a)
b = dbinom(x-1,n-1,p0); b; sum(b)

a[1]; a[k]
b[1]; b[k]
g1 = ((1-alpha)*(b[k]-a[k])-(b[k]*sum(a)-a[k]*sum(b)))/(b[1]*a[k]-a[1]*b[k]); g1
g2 = (sum(a)-g1*a[1]-(1-alpha))/a[k]; g2

sum(a)-g1*a[1]-g2*a[k]
sum(b)-g1*b[1]-g2*b[k]

Power = c()
p_v = seq(0.001,1,0.001)
i = 1
for (p in p_v) {


     E1 = sum(dbinom(c((c1-1),(c2+1):10),n,p))
     E2 = g1*dbinom(c1,n,p)
     E3 = g2*dbinom(c2,n,p)
     Power[i] = E1+E2+E3
     i = i+1

}

plot(p_v,Power,ylim=c(0,1),xlim=c(0,1),type="l",xlab="p",ylab="Power
function",lwd=2)
abline(h=alpha,lty=2)
text(0.8,0.1,expression(alpha*" = 0.05"),cex=1.5)
```