Homework#1 Statistical Inference III

Name: Shih Jia-Han

Problem 1

Show that $C = \{ \phi, (-\infty, a), [a, \infty), Z \}$ is a sigma-field, where $Z = (-\infty, \infty)$.

Proof.

The properties of sigma-field are verified as follows:

1. $\phi \in C$.

2. $\phi \in C \Rightarrow \phi^c = Z \in C$.

$Z \in C \Rightarrow Z^c = \phi \in C$.

$(-\infty, a) \in C \Rightarrow (-\infty, a)^c = [a, \infty) \in C$.

$[a, \infty) \in C \Rightarrow [a, \infty)^c = (-\infty, a) \in C$.

3. Consider a sequence $c_1, c_2, \cdots \in C$.

I. If $c_i = Z$ for some i then $\bigcup_{i=1}^{\infty} c_i = Z$.

II. If $c_i = \phi$ for all i then $\bigcup_{i=1}^{\infty} c_i = \phi$.

III. If $c_i = (-\infty, a)$ for some i, $c_i \neq [a, \infty)$ for all i and $c_i \neq Z$ for all i then $\bigcup_{i=1}^{\infty} c_i = (-\infty, a)$.

IV. If $c_i = [a, \infty)$ for some i, $c_i \neq (-\infty, a)$ for all i and $c_i \neq Z$ for all i then $\bigcup_{i=1}^{\infty} c_i = [a, \infty)$.

V. If $c_i = (-\infty, a)$ for some i, $c_i = [a, \infty)$ for some i and $c_i \neq Z$ for all i then $\bigcup_{i=1}^{\infty} c_i = Z$.

Thus, we have shown that C is a sigma-field. □

Problem 2

Show that if $X \sim N(0,1)$ then $Y = X^2 \sim \chi^2_{\mathrm{df}=1}$.

Proof.

The probability density function of X is

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \qquad -\infty < x < \infty.$$

If $0 < x < \infty$, consider the change of variable

$$y = x^2 \Rightarrow x = \sqrt{y} \Rightarrow |J| = \left|\frac{dx}{dy}\right| = \frac{1}{2} y^{-\frac{1}{2}}.$$

Then we have

$$f_Y^+(y) = f_X(\sqrt{y})\,|J| = \frac{1}{2\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}.$$

Similarly, if $-\infty < x < 0$,

$$y = x^2 \Rightarrow x = -\sqrt{y} \Rightarrow |J| = \left|\frac{dx}{dy}\right| = \frac{1}{2} y^{-\frac{1}{2}}.$$

Then we have

$$f_Y^-(y) = f_X(-\sqrt{y})\,|J| = \frac{1}{2\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}.$$

Therefore, we obtain

$$f_Y(y) = f_Y^+(y) + f_Y^-(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}.$$

That is $Y = X^2$ follows chi-square distribution with degree of freedom equal to 1. Hence we finished the proof. □

Another proof.

The cumulative distribution function of $Y = X^2$ is

$$\begin{aligned} F_Y(y) &= \Pr(Y \le y) = \Pr(X^2 \le y) \\ &= \Pr(-\sqrt{y} \le X \le \sqrt{y}) \\ &= \Pr(X \le \sqrt{y}) - \Pr(X \le -\sqrt{y}) \\ &= \Phi(\sqrt{y}) - \Phi(-\sqrt{y}), \end{aligned}$$

where $\Phi(\cdot)$ is the cumulative distribution function of standard normal distribution. Thus, the probability density function of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy} F_Y(y) = \frac{d}{dy}\{\Phi(\sqrt{y}) - \Phi(-\sqrt{y})\} \\ &= \frac{d}{dy}\Phi(\sqrt{y}) - \frac{d}{dy}\Phi(-\sqrt{y}) \\ &= \frac{1}{2} y^{-\frac{1}{2}} \phi(\sqrt{y}) + \frac{1}{2} y^{-\frac{1}{2}} \phi(-\sqrt{y}) \\ &= \frac{1}{2}\frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} + \frac{1}{2}\frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}} \\ &= \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}, \end{aligned}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

is the probability density function of standard normal distribution. As a result, we have obtained

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} y^{-\frac{1}{2}} e^{-\frac{y}{2}}.$$

That is $Y = X^2$ follows chi-square distribution with degree of freedom equal to 1. Hence we finished the proof. □

Problem 3

Show the following inequality

$$\left|\frac{e^{az}-1}{z}\right| \le \frac{e^{\delta|a|}}{\delta}, \qquad \text{for all } |z| \le \delta .$$

Proof.

For all $|z| \le \delta$, and since exponential function can be defined by power series, we have

$$\begin{aligned}\left|\frac{e^{az}-1}{z}\right| &= \left|\frac{1}{z}\left\{\sum_{i=0}^{\infty}\frac{(az)^i}{i!}-1\right\}\right| = \left|\frac{1}{z}\left\{1+\sum_{i=1}^{\infty}\frac{(az)^i}{i!}-1\right\}\right| = \left|\frac{1}{z}\sum_{i=1}^{\infty}\frac{a^i z^i}{i!}\right| = \left|\sum_{i=1}^{\infty}\frac{a^i z^{i-1}}{i!}\right| \\ &\le \sum_{i=1}^{\infty}\frac{|a|^i |z|^{i-1}}{i!} \le \sum_{i=1}^{\infty}\frac{|a|^i \delta^{i-1}}{i!} = \frac{1}{\delta}\sum_{i=1}^{\infty}\frac{|a|^i \delta^i}{i!} = \frac{e^{\delta|a|}-1}{\delta} \\ &\le \frac{e^{\delta|a|}}{\delta}.\end{aligned}$$

Then we have proven the desired result. □

For further discussion, we define

$$M_a(z) = \left|\frac{e^{az}-1}{z}\right|.$$

The inequality above is equivalent to

$$0 \le M_a(z) \le \frac{e^{\delta|a|}}{\delta}, \qquad \text{for all } |z| \le \delta .$$

There are three different cases including a is positive, negative or zero. The case of a is equal to zero is trivial since the M function becomes zero. For the other two cases, it is possible to plot the M function and its bounds to actually see how these bounds work. The results are given in Figure 1 and 2.

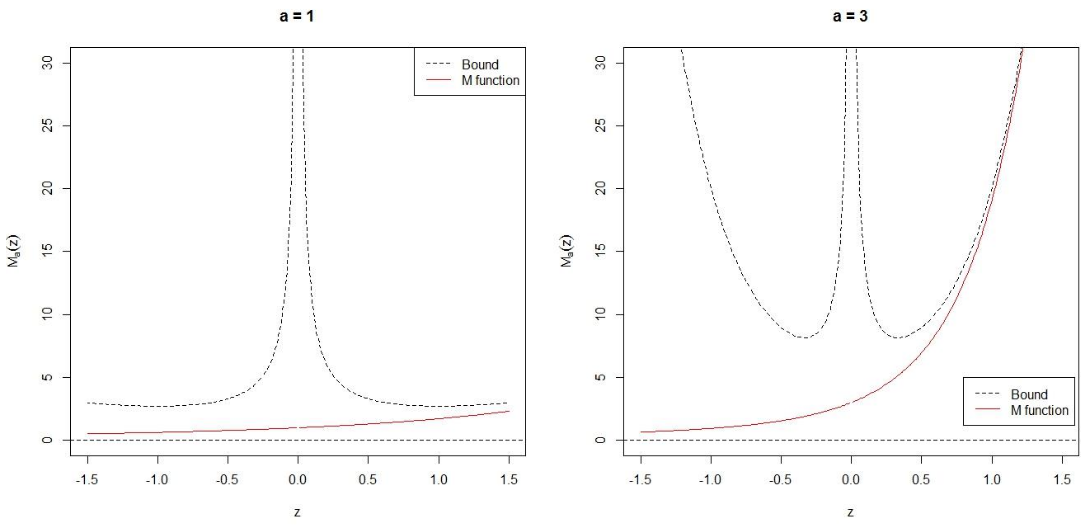

Figure 1. The plots of M functions and corresponding bounds when a is positive.

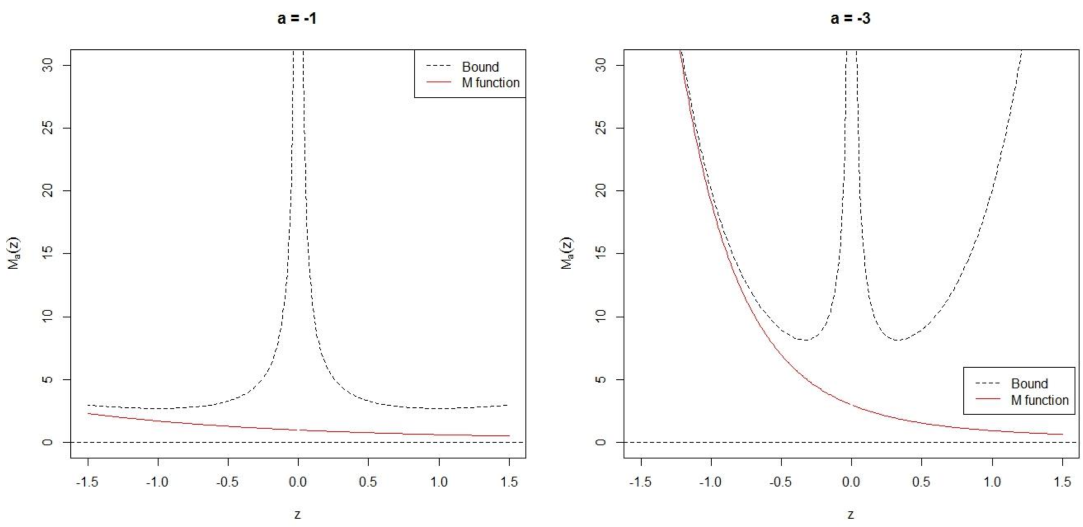

Figure 2. The plots of M functions and corresponding bounds when a is negative.