Homework#2 Statistical Inference II

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Problem 2.1

Referring to Example 1.5, suppose that X has the binomial distribution $\text{Bin}(n, p)$ and the family of prior distribution for p is the family of beta distribution $\text{Beta}(a, b)$.

(a) Show that the marginal distribution of X is the *beta-binomial* distribution with mass function

$$\binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(x+a)\Gamma(n-x+b)}{\Gamma(n+a+b)}.$$

Solution:

The density function of $X \mid p \sim \text{Bin}(n, p)$ and $p \sim \text{Beta}(a, b)$ are

$$f(x \mid p) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \cdots, n$$

and

$$\pi(p) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1}, \quad 0 < p < 1.$$

Therefore, the joint density function is

$$\begin{aligned} f(x, p) &= \binom{n}{x} p^x (1-p)^{n-x} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{a-1}(1-p)^{b-1} \\ &= \binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} p^{x+a-1}(1-p)^{n-x+b-1}. \end{aligned}$$

Then we obtain the marginal distribution of X by integrating the joint density function

$$f(x)=\int_0^1 f(x,p)dp=\binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 p^{x+a-1}(1-p)^{n-x+b-1}dp$$
$$=\binom{n}{x}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(x+a)\Gamma(n-x+b)}{\Gamma(n+a+b)}.$$

(b) Show that the mean and the variance of the beta-binomial is given by

$$E(X)=\frac{na}{a+b} \quad \text{and} \quad \text{var}(X)=n\left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)\left(\frac{a+b+n}{a+b+1}\right).$$

Solution:

It is easy to obtain

$$E(X)=E\{E(X\mid p)\}=E(np)=nE(p)=\frac{na}{a+b}.$$

Similarly,

$$\text{var}(X)=\text{var}\{E(X\mid p)\}+E\{\text{var}(X\mid p)\}$$
$$=\text{var}(np)+E\{np(1-p)\}=n^2\,\text{var}(p)+nE\{p(1-p)\}$$
$$=n^2\frac{ab}{(a+b)^2(a+b+1)}+n\int_0^1\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}p^a(1-p)^b dp$$
$$=n^2\frac{ab}{(a+b)^2(a+b+1)}+n\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)}$$
$$=n^2\frac{ab}{(a+b)^2(a+b+1)}+n\frac{ab}{(a+b)(a+b+1)}$$
$$=n\left(\frac{a}{a+b}\right)\left(\frac{b}{a+b}\right)\left(\frac{a+b+n}{a+b+1}\right).$$

Problem 3.1 [p.285]

For the situation of Example 3.1:

(a) Verify the Bayes estimator will only depend on the data through $Y = \max_i X_i$.

Solution:

According to Example 3.1, we have

$$X_i \mid \theta \sim U(0, \theta), \quad 0 \le x_i \le \theta \quad \text{for} \quad i = 1, 2, \cdots, n$$

and

$$1/\theta \sim \text{Gamma}(a, b), \quad 1/\theta > 0, \text{ where } a, b \text{ are known.}$$

Therefore, the joint density of X is

$$\begin{aligned} f(x_1, x_2, \cdots, x_n \mid \theta) &= \prod_{i=1}^{n} f(x_i \mid \theta) = \prod_{i=1}^{n} \frac{1}{\theta} I(0 \le x_i \le \theta) \\ &= \frac{1}{\theta^n} I(0 \le x_1, x_2, \cdots, x_n \le \theta) \\ &= \frac{1}{\theta^n} I(\max_i x_i \le \theta \le \infty) = \frac{1}{\theta^n} I(y \le \theta \le \infty). \end{aligned}$$

Here we define $z = 1/\theta$ and make a transformation

$$\theta = \frac{1}{z} \Rightarrow z = \frac{1}{\theta} \Rightarrow \left| \frac{dz}{d\theta} \right| = \theta^{-2}.$$

Then the density function of θ is

$$\pi_\theta(\theta \mid a, b) = \pi_z(1/\theta \mid a, b)\theta^{-2} = \frac{1}{\Gamma(a) b^a \theta^{a+1}} e^{-\frac{1}{b\theta}}.$$

This is the density function of inverse gamma distribution.

Now we can derive the posterior distribution

$$\pi_\theta(\theta \mid x) \propto \frac{1}{\theta^{a+1}} e^{-\frac{1}{b\theta}} \frac{1}{\theta^n} I(y \le \theta \le \infty)$$
$$\propto \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} I(y \le \theta \le \infty).$$

Therefore, the Bayes estimator is

$$\delta_\Lambda = E\{\pi_\theta(\theta \mid x, a, b)\} = \frac{\int_0^\infty \theta \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} I(y \le \theta \le \infty) d\theta}{\int_0^\infty \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} I(y \le \theta \le \infty) d\theta} = \frac{\int_y^\infty \frac{1}{\theta^{n+a}} e^{-\frac{1}{b\theta}} d\theta}{\int_y^\infty \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} d\theta}.$$

Hence the Bayes estimator is only depend on $Y = \max_i X_i$.

(b) Show that $E\{\pi_\theta(\theta \mid x, a, b)\}$ can be expressed as

$$E\{\pi_\theta(\theta \mid x, a, b)\} = \frac{1}{b(n+a-1)} \frac{\Pr(\chi^2_{2(n+a-1)} < 2/by)}{\Pr(\chi^2_{2(n+a)} < 2/by)},$$

where χ^2_ν is a chi-squared random variable with ν degrees of freedom. (In this form, the estimator is particularly easy to calculate, as many packages will have the chi-squared distribution built in.)

Solution:

By (a), we have

$$E\{\pi_\theta(\theta \mid x, a, b)\} = \frac{\int_y^\infty \frac{1}{\theta^{n+a}} e^{-\frac{1}{b\theta}} d\theta}{\int_y^\infty \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} d\theta}.$$

First, we consider a transformation

$$s = \frac{2}{b\theta} \Rightarrow ds = -\frac{2}{b\theta^2}\, d\theta \quad .$$

We apply this transformation to the numerator, we have

$$\int_y^\infty \frac{1}{\theta^{n+a}} e^{-\frac{1}{b\theta}} d\theta = -\frac{b}{2}\int_y^\infty \frac{1}{\theta^{n+a-2}} e^{-\frac{1}{b\theta}} \left(-\frac{2}{b\theta^2}\right) d\theta = -\frac{b}{2}\int_{2/by}^{0} \left(\frac{b}{2}\right)^{n+a-2} s^{n+a-2} e^{-\frac{s}{2}} ds$$
$$= b^{n+a-1}\Gamma(n+a-1)\int_0^{2/by} \frac{1}{\Gamma(n+a-1)2^{n+a-1}} s^{n+a-2} e^{-\frac{s}{2}} ds$$
$$= b^{n+a-1}\Gamma(n+a-1)\Pr(\chi^2_{2(n+a-1)} < 2/by).$$

Similarly, we apply this transformation to the denominator, we obtain

$$\int_y^\infty \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} d\theta = b^{n+a}\Gamma(n+a)\Pr(\chi^2_{2(n+a)} < 2/by).$$

Therefore, the Bayes estimator can be expressed as

$$E\{\pi_\theta(\theta \mid x, a, b)\} = \frac{\int_y^\infty \frac{1}{\theta^{n+a}} e^{-\frac{1}{b\theta}} d\theta}{\int_y^\infty \frac{1}{\theta^{n+a+1}} e^{-\frac{1}{b\theta}} d\theta} = \frac{b^{n+a-1}\Gamma(n+a-1)\Pr(\chi^2_{2(n+a-1)} < 2/by)}{b^{n+a}\Gamma(n+a)\Pr(\chi^2_{2(n+a)} < 2/by)}$$
$$= \frac{1}{b(n+a-1)} \frac{\Pr(\chi^2_{2(n+a-1)} < 2/by)}{\Pr(\chi^2_{2(n+a)} < 2/by)}.$$

Problem 3.9

For the natural exponential family $p_\eta(x)$ of (3.3.18) and the conjugate prior $\pi(\eta\,|\,k,\mu)$ of (3.3.19) establish that:

(a) $E(X) = A'(\eta)$ and $\text{var}(X) = A''(\eta)$, where the expectation is with respect to the sampling density $p_\eta(x)$.

Solution:

The natural exponential family $p_\eta(x)$ of (3.3.18) is

$$p_\eta(x) = e^{\eta x - A(\eta)} h(x), \quad -\infty < x < \infty.$$

Since it is a density function, we have

$$\int_{-\infty}^{\infty} p_\eta(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx = 1 \Rightarrow \frac{\partial}{\partial \eta}\int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial}{\partial \eta} e^{\eta x - A(\eta)} h(x)dx = 0 \Rightarrow \int_{-\infty}^{\infty} \{x - A'(\eta)\} e^{\eta x - A(\eta)} h(x)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} x e^{\eta x - A(\eta)} h(x)dx = A'(\eta)\int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx \Rightarrow E(X) = A'(\eta).$$

Similarly,

$$\int_{-\infty}^{\infty} p_\eta(x)dx = 1 \Rightarrow \int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx = 1 \Rightarrow \frac{\partial^2}{\partial \eta^2}\int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{\partial^2}{\partial \eta^2} e^{\eta x - A(\eta)} h(x)dx = 0 \Rightarrow \int_{-\infty}^{\infty} \frac{\partial}{\partial \eta}\{x - A'(\eta)\} e^{\eta x - A(\eta)} h(x)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} [-A''(\eta) + \{x - A'(\eta)\}^2] e^{\eta x - A(\eta)} h(x)dx = 0$$

$$\Rightarrow \int_{-\infty}^{\infty} \{x - A'(\eta)\}^2 e^{\eta x - A(\eta)} h(x)dx = A''(\eta)\int_{-\infty}^{\infty} e^{\eta x - A(\eta)} h(x)dx \Rightarrow \text{var}(X) = A''(\eta).$$

Hence we have shown

$$E(X) = A'(\eta) \text{ and } \operatorname{var}(X) = A''(\eta).$$

(b) $E\{A'(\eta)\} = \mu$ and $\operatorname{var}\{A'(\eta)\} = E\{A''(\eta)\}/k$, where the expectation is with respect to the prior distribution.

Solution:

The prior distribution of (3.3.19) is

$$\pi(\eta \mid k, \mu) = c(k, \mu) e^{k\eta\mu - kA(\eta)}.$$

Suppose the support of η is (a, b), then $e^{k\eta\mu - kA(\eta)} \to 0$ as $\eta \to a$ or b. Let

$$u = c(k, \mu) e^{k\eta\mu} \Rightarrow du = c(k, \mu) k\mu e^{k\eta\mu} d\eta, \quad v = \frac{-1}{k} e^{-kA(\eta)} \Rightarrow dv = A'(\eta) e^{-kA(\eta)} d\eta.$$

Therefore, by the method of integration by parts, we have

$$\begin{aligned}
E\{A'(\eta)\} &= \int A'(\eta) c(k, \mu) e^{k\eta\mu - kA(\eta)} d\eta \\
&= \int u dv \\
&= [uv]_a^b - \int v du \\
&= \left[\frac{-1}{k} c(k, \mu) e^{k\eta\mu - kA(\eta)} \right]_a^b + \int \frac{1}{k} \cdot k\mu c(k, \mu) e^{k\eta\mu - kA(\eta)} d\eta \\
&= \mu \int c(k, \mu) e^{k\eta\mu - kA(\eta)} d\eta \\
&= \mu.
\end{aligned}$$

Hence we obtain the result

$$E\{A'(\eta)\} = \mu.$$

We have

$$
\begin{aligned}
\operatorname{var}\{A'(\eta)\} &= E[\{A'(\eta)-\mu\}^2] \\
&= \int \{A'(\eta)-\mu\}^2 c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta \\
&= \int \{A'(\eta)^2+\mu^2-2A'(\eta)\mu\}c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta \\
&= \int A'(\eta)^2 c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta+\mu^2-2\mu^2 \\
&= \int \left\{A'(\eta)^2-\frac{1}{k}A''(\eta)\right\}c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta-\mu^2+\frac{E\{A''(\eta)\}}{k}.
\end{aligned}
$$

Let

$$u=c(k,\mu)e^{k\eta\mu} \Rightarrow du=c(k,\mu)k\mu e^{k\eta\mu}d\eta\,,$$

$$v=-\frac{1}{k}A'(\eta)e^{-kA(\eta)} \Rightarrow dv=\left(A'(\eta)^2-\frac{1}{k}A''(\eta)\right)e^{-kA(\eta)}d\eta\,.$$

Therefore, by the method of integration by parts, we have

$$
\begin{aligned}
&\int \left\{A'(\eta)^2-\frac{1}{k}A''(\eta)\right\}c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=\int u\,dv-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=[uv]_a^b-\int v\,du-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=\left[-\frac{1}{k}A'(\eta)c(k,\mu)e^{k\eta\mu-kA(\eta)}\right]_a^b \\
&\qquad\qquad +\int \frac{1}{k}\cdot k\mu A'(\eta)c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=\mu\int A'(\eta)c(k,\mu)e^{k\eta\mu-kA(\eta)}d\eta-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=\mu^2-\mu^2+\frac{E\{A''(\eta)\}}{k} \\
&=\frac{E\{A''(\eta)\}}{k}.
\end{aligned}
$$

Hence we obtain the result

$$\operatorname{var}\{A'(\eta)\}=E\{A''(\eta)\}/k$$

Problem 6.4 [p.295]

(a)

Let

$$X_i \mid \theta \sim N(\theta, \sigma^2)\,,\quad \sigma^2 \text{ is known and } \theta \sim N(0, \tau^2)\,,\quad \tau^2 \text{ is unknown.}$$

Then the marginal distribution (6.6.4) is

$$
\begin{aligned}
& m(x \mid \tau^2) \\
&= \int_{-\infty}^{\infty} \prod_{i=1}^{p} f(x_i \mid \theta)\pi(\theta)\,d\theta \\
&= \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{2\pi\sigma^2}}\right)^{p} \frac{1}{\sqrt{2\pi\tau^2}} \exp\left\{-\sum_{i=1}^{p} \frac{(x_i-\theta)^2}{2\sigma^2} - \frac{\theta^2}{2\tau^2}\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p+1} \frac{1}{\sigma^p \tau} \int_{-\infty}^{\infty} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{p} x_i^2 + \frac{\theta}{\sigma^2}\sum_{i=1}^{p} x_i - \frac{p\theta^2}{2\sigma^2} - \frac{\theta^2}{2\tau^2}\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p+1} \frac{1}{\sigma^p \tau} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{p} x_i^2\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left[\left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)\theta^2 - \frac{2p\bar{x}}{\sigma^2}\right]\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p+1} \frac{1}{\sigma^p \tau} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{p} x_i^2\right\} \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\theta^2 - \frac{2\bar{x}p/\sigma^2}{p/\sigma^2+1/\tau^2}\theta\right)\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p+1} \frac{1}{\sigma^p \tau} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{p} x_i^2 + \frac{\bar{x}^2 p^2/\sigma^4}{2(p/\sigma^2+1/\tau^2)}\right\} \\
&\qquad\qquad \times \int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2}\left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\theta - \frac{\bar{x}p/\sigma^2}{p/\sigma^2+1/\tau^2}\right)^2\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p} \frac{1}{\sigma^p \tau} \exp\left\{\frac{-1}{2\sigma^2}\sum_{i=1}^{p} x_i^2 + \frac{\bar{x}^2 p^2/\sigma^4}{2(p/\sigma^2+1/\tau^2)}\right\} \left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)^{-\frac{1}{2}} \\
&\qquad\qquad \times \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}\left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\frac{p}{\sigma^2}+\frac{1}{\tau^2}\right)\left(\theta - \frac{\bar{x}p/\sigma^2}{p/\sigma^2+1/\tau^2}\right)^2\right\} d\theta \\
&= \left(\frac{1}{\sqrt{2\pi}}\right)^{p} \frac{1}{\sigma^p \tau} \left(\frac{\sigma^2\tau^2}{p\tau^2+\sigma^2}\right)^{\frac{1}{2}} \exp\left\{\frac{-1}{2\sigma^2}\left(\sum_{i=1}^{p} x_i^2 - p\bar{x}^2\right) - \frac{p\bar{x}^2}{2\sigma^2} + \frac{\bar{x}^2 p^2/\sigma^4}{2(p/\sigma^2+1/\tau^2)}\right\}
\end{aligned}
$$

$$= \left(\frac{1}{2\pi\sigma} \right)^{\frac{p}{2}} \left(\frac{\sigma^2}{p\tau^2 + \sigma^2} \right)^{\frac{1}{2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{p} (x_i - \bar{x})^2 + \frac{\bar{x}^2 p^2 / \sigma^2 - \bar{x}^2 p^2 / \sigma^2 - \bar{x}^2 p / \tau^2}{2\sigma^2 (p / \sigma^2 + 1 / \tau^2)} \right\}$$

$$= \left(\frac{1}{2\pi\sigma} \right)^{\frac{p}{2}} \left(\frac{\sigma^2}{p\tau^2 + \sigma^2} \right)^{\frac{1}{2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{p} (x_i - \bar{x})^2 - \frac{\bar{x}^2 p / \tau^2}{2\sigma^2 (p / \sigma^2 + 1 / \tau^2)} \right\}$$

$$= \left(\frac{1}{2\pi\sigma} \right)^{\frac{p}{2}} \left(\frac{\sigma^2}{p\tau^2 + \sigma^2} \right)^{\frac{1}{2}} \exp\left\{ \frac{-1}{2} \left(\sum_{i=1}^{p} \frac{(x_i - \bar{x})^2}{\sigma^2} + \frac{p\bar{x}^2}{p\tau^2 + \sigma^2} \right) \right\}.$$

Hence we obtain the marginal distribution

$$m(x \mid \tau^2) = \left(\frac{1}{2\pi\sigma} \right)^{\frac{p}{2}} \left(\frac{\sigma^2}{p\tau^2 + \sigma^2} \right)^{\frac{1}{2}} \exp\left\{ \frac{-1}{2} \left(\sum_{i=1}^{p} \frac{(x_i - \bar{x})^2}{\sigma^2} + \frac{p\bar{x}^2}{p\tau^2 + \sigma^2} \right) \right\}.$$