Report#1 Statistical Inference I

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Problem 5.1 (p.65)

Determined the natural parameter space of (5.2) when $s=1$, $T_1(x)=x$, μ is Lebesgue measure and $h(x)$ is (i) $e^{-|x|}$ and (ii) $e^{-|x|}/(1+x^2)$.

Solution of (i):

The Equation (5.2) is

$$p(x|\eta)=\exp\left\{\sum_{i=1}^{s}\eta_i T_i(x)-A(\eta)\right\}h(x).$$

Let $s=1$, $T_1(x)=x$ and $h(x)=e^{-|x|}$ then we obtain

$$p(x|\eta)=\exp\{\eta_1 x-|x|-A(\eta)\}.$$

Since $p(x|\eta)$ is a density function and μ is Lebesgue measure. We have

$$\int_\chi \exp\{\eta_1 x-|x|-A(\eta)\}=1$$
$$\Rightarrow \int_\chi \exp\{\eta_1 x-|x|\}=e^{A(\eta)}<\infty,$$

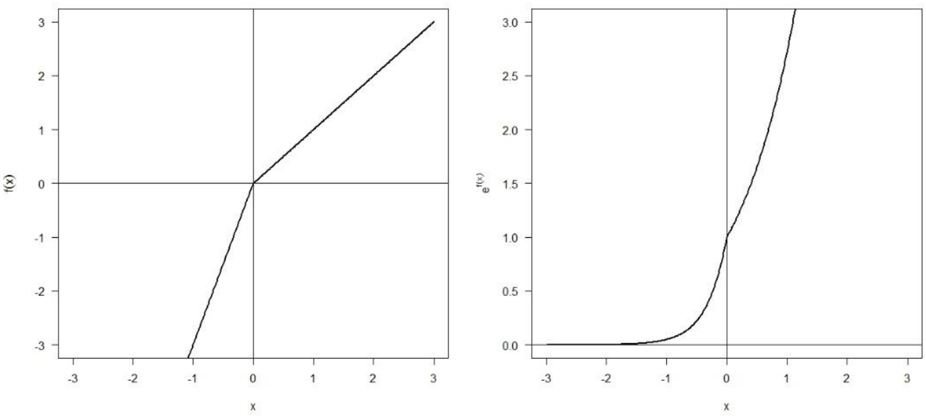
where χ is the sample space.

Consider the function

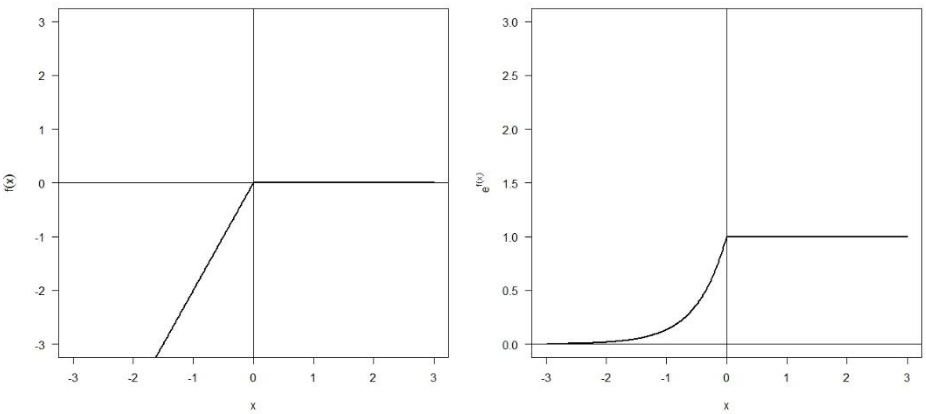
$$f(x)=\eta_1 x-|x|$$
$$=\begin{cases}(\eta_1-1)x & \text{if } x\ge 0\\ (\eta_1+1)x & \text{if } x<0\end{cases}.$$

Therefore, we can separate the parameter space into five different cases. Then we can plot graphs of $f(x)$ and $\exp\{f(x)\}$ by R in different cases to study the relationship between the sample space and the parameter space.

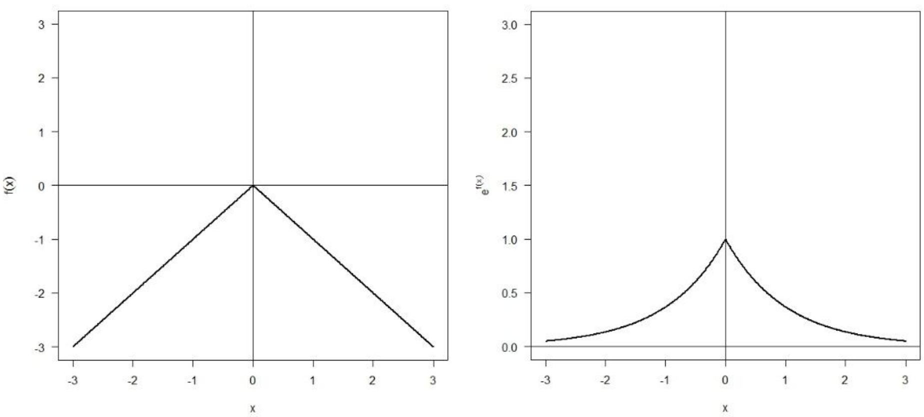
Case 1 $(\eta > 1)$: Example: $\eta = 2$



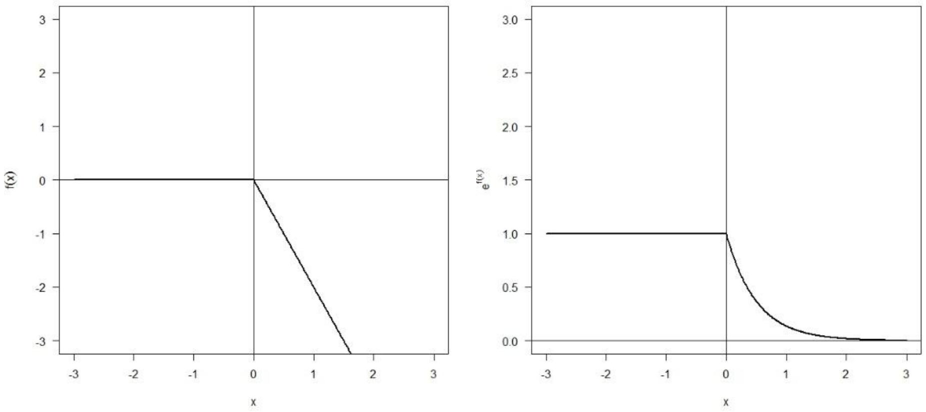
Case 2 $(\eta = 1)$:



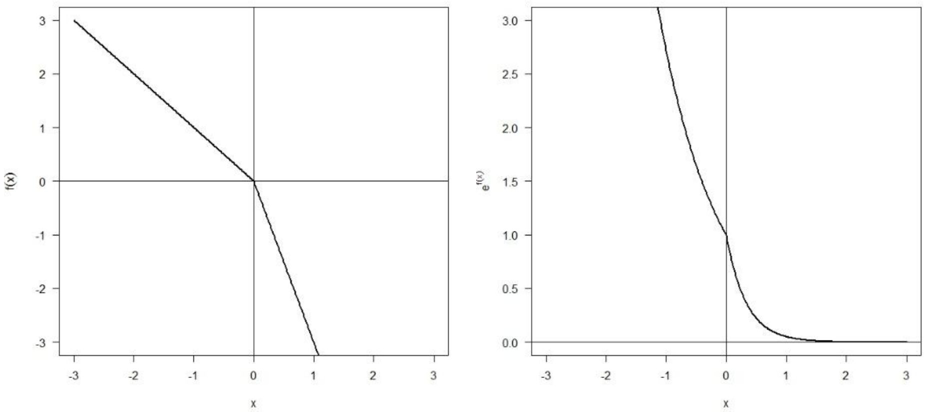
Case 3 $(-1 < \eta < 1)$: Example: $\eta = 0$



Case 4 $(\eta = -1)$:



Case 5 $(\eta < -1)$: Example: $\eta = -2$



Therefore, we have the following results:

If the sample space $\chi = (-\infty, \infty)$, the natural parameter space is $\Theta = (-1, 1)$.

If the sample space $\chi = [0, \infty)$, the natural parameter space is $\Theta = (-\infty, 1)$.

If the sample space $\chi = (-\infty, 0]$, the natural parameter space is $\Theta = (-1, \infty)$.

Solution of (ii):

Similarly, the Equation (5.2) is

$$p(x|\eta) = \exp\left\{\sum_{i=1}^{s} \eta_i T_i(x) - A(\eta)\right\} h(x).$$

Let $s = 1$, $T_1(x) = x$ and $h(x) = e^{-|x|}/(1+x^2)$ then we obtain

$$p(x|\eta) = \exp\{\eta_1 x - |x| - \log(1+x^2) - A(\eta)\}.$$

Since $p(x|\eta)$ is a density function and μ is Lebesgue measure. We have

$$\int_{\chi} \exp\{\eta_1 x - |x| - \log(1+x^2) - A(\eta)\} = 1$$
$$\Rightarrow \int_{\chi} \exp\{\eta_1 x - |x| - \log(1+x^2)\} = e^{A(\eta)} < \infty,$$

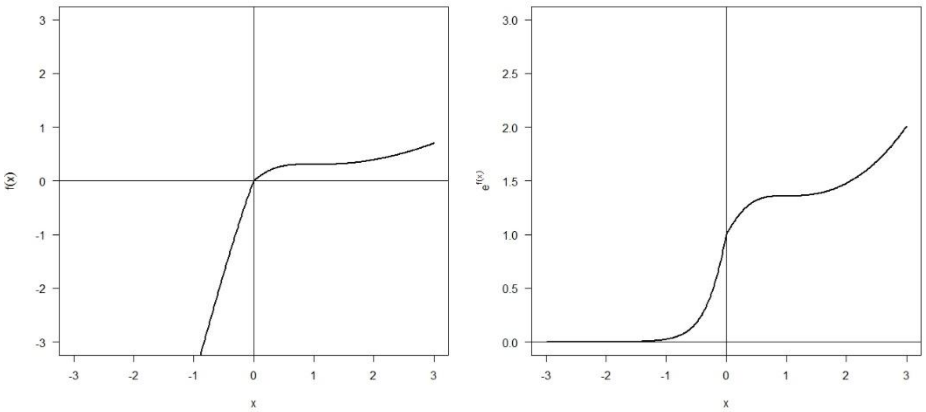
where χ is the sample space.

Consider the function

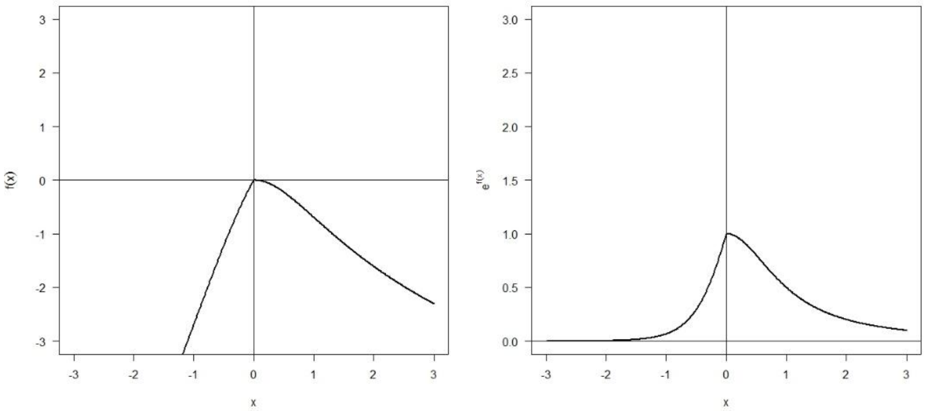
$$f(x) = \eta_1 x - |x| - \log(1+x^2)$$
$$= \begin{cases} (\eta_1 - 1)x - \log(1+x^2) & \text{if } x \geq 0 \\ (\eta_1 + 1)x - \log(1+x^2) & \text{if } x < 0 \end{cases}.$$

Therefore, we can separate the parameter space into five different cases. Then we can plot graphs of $f(x)$ and $\exp\{f(x)\}$ by R in different cases to study the relationship between the sample space and the parameter space.

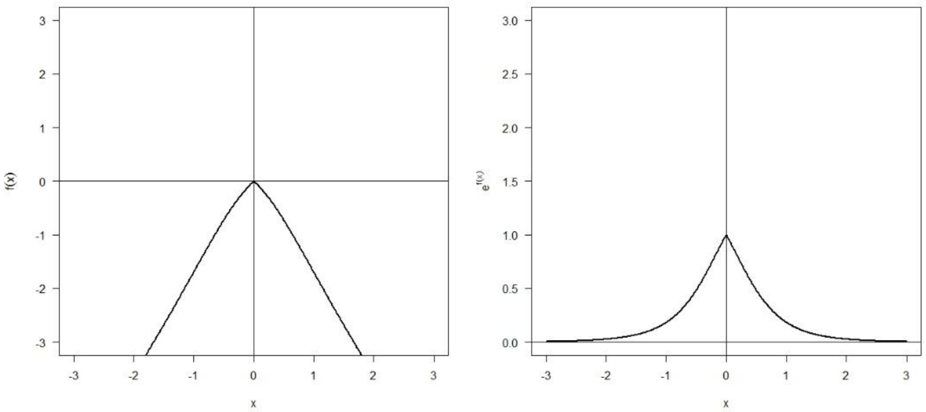
Case 1 $(\eta > 1)$: Example: $\eta = 2$



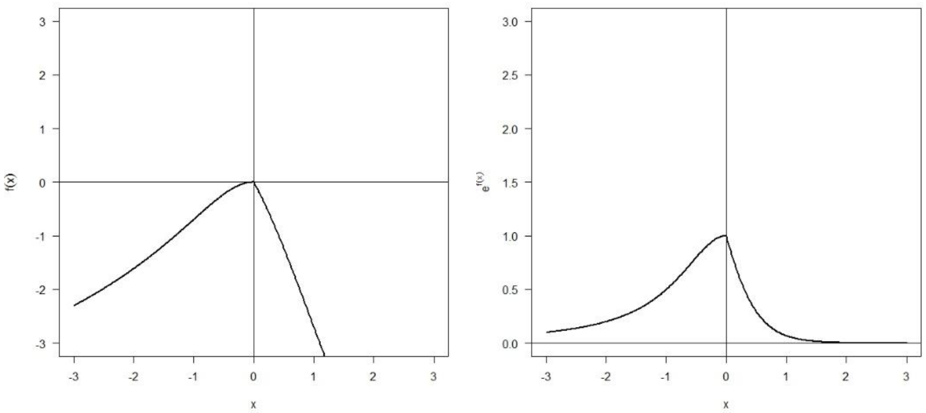
Case 2 $(\eta = 1)$:



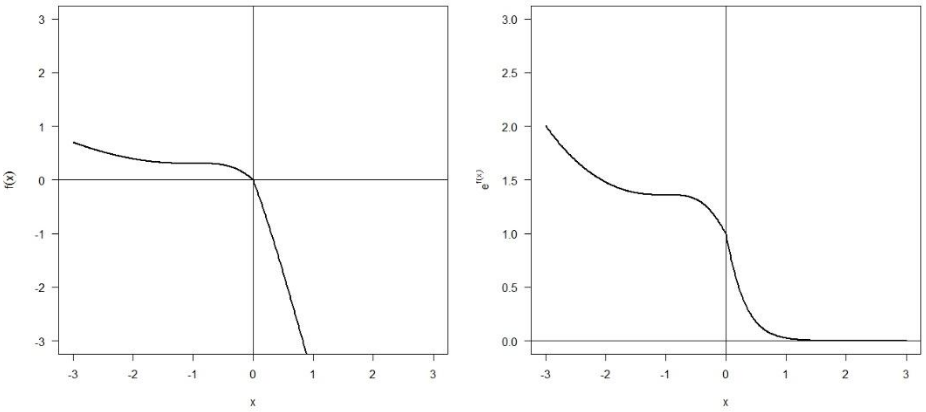
Case 3 $(-1 < \eta < 1)$: Example: $\eta = 0$



Case 4 $(\eta = -1)$:



Case 5 $(\eta < -1)$: Example: $\eta = -2$



Therefore, we have the following results:

If the sample space $\chi = (-\infty, \infty)$, the natural parameter space is $\Theta = [-1, 1]$.

If the sample space $\chi = [0, \infty)$, the natural parameter space is $\Theta = (-\infty, 1]$.

If the sample space $\chi = (-\infty, 0]$, the natural parameter space is $\Theta = [-1, \infty)$.

R code

```
##### 5.1 (i) #####

f1=function(eta,x) {

    eta*x-abs(x)

}

eta=2

q=seq(-3,3,0.01)
plot(q,f1(eta,q),xlim=c(-3,3),ylim=c(-3,3),type="l",xlab=expression(x),ylab=expressi
on(f(x)),las=1,lwd=2)
abline(h=0)
abline(v=0)

plot(q,exp(f1(eta,q)),xlim=c(-3,3),ylim=c(0,3),type="l",xlab=expression(x),ylab=expr
ession(e^f(x)),las=1,lwd=2)
abline(h=0)
abline(v=0)
```

```
##### 5.1 (ii) #####
f2=function(eta,x) {

        eta*x-abs(x)-log(1+x^2)

}

eta=2

q=seq(-3,3,0.01)
plot(q,f2(eta,q),xlim=c(-3,3),ylim=c(-3,3),type="l",xlab=expression(x),ylab=expressi
on(f(x)),las=1,lwd=2)
abline(h=0)
abline(v=0)

plot(q,exp(f2(eta,q)),xlim=c(-3,3),ylim=c(0,3),type="l",xlab=expression(x),ylab=expr
ession(e^f(x)),las=1,lwd=2)
abline(h=0)
abline(v=0)
```