Final exam, Statistical Inference I: Date 1/11 (2015 Fall): [+40points]

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Q1 [+18] Let $X_1,..,X_n \overset{iid}{\sim} N(\xi,\sigma^2)$, where $\theta=(\xi,\sigma^2)$. Let $\overline{X}=\sum_{i=1}^n X_i/n$, and $S^2=\sum_{i=1}^n (X_i-\overline{X})^2$.

a) **[+2]** Derive the UMVUE of ξ^2 (with proof)

<u>Ans:</u>

Since $\left(\overline{X}=\frac{1}{n}\sum_{i=1}^n X_i,\ S^2=\sum_{i=1}^n (X_i-\overline{X})^2\right)$ is a complete sufficient statistic for $\theta=(\xi,\sigma^2)$.

Their distribution are

$$\overline{X}\sim N\left(\xi,\frac{\sigma^2}{n}\right) \text{ and } \frac{S^2}{\sigma^2}\sim \chi^2_{df=n-1}.$$

Then we have

$$E(\overline{X}^2)=E(\overline{X})^2+\mathrm{var}(\overline{X})=\xi^2+\frac{\sigma^2}{n}$$

and

$$E\left(\frac{S^2}{\sigma^2}\right)=n-1\Rightarrow E\left(\frac{S^2}{n(n-1)}\right)=\frac{\sigma^2}{n}.$$

Therefore,

$$E\left(\overline{X}^2-\frac{S^2}{n(n-1)}\right)=E(\overline{X}^2)-E\left(\frac{S^2}{n(n-1)}\right)=\xi^2+\frac{\sigma^2}{n}-\frac{\sigma^2}{n}=\xi^2.$$

Since $\overline{X}^2-\frac{S^2}{n(n-1)}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^2-\frac{S^2}{n(n-1)}$ is the UMVUE of ξ^2.

b) **[+4]** For the above estimator, derive the risk $R(\theta,\delta)$ under the squared error loss (include calculation details).

Ans:

Since $\overline{X}$ is complete sufficient statistic for ξ and S^2 is ancillary for ξ. By Basu's Theorem, $\overline{X}$ and S^2 are independent.

$$R(\theta,\delta)=E\{L(\theta,\delta)\}=E\left\{\left(\overline{X}^2-\frac{S^2}{n(n-1)}\right)-\xi^2\right\}^2=\mathrm{var}\left(\overline{X}^2-\frac{S^2}{n(n-1)}\right)$$
$$=\mathrm{var}(\overline{X}^2)+\mathrm{var}\left(\frac{S^2}{n(n-1)}\right).$$

For the following computation, I directly use the formula of $E(\overline{X}^k)$. This formula will be proved latter.

$$\mathrm{var}(\overline{X}^2)=E(\overline{X}^4)-\{E(\overline{X}^2)\}^2=\xi^4+\frac{6\xi^2\sigma^2}{n}+\frac{3\sigma^4}{n^2}-\left\{\xi^2+\frac{\sigma^2}{n}\right\}^2$$
$$=\xi^4+\frac{6\xi^2\sigma^2}{n}+\frac{3\sigma^4}{n^2}-\xi^4-\frac{\sigma^4}{n^2}-\frac{2\xi^2\sigma^2}{n}$$
$$=\frac{4\xi^2\sigma^2}{n}+\frac{2\sigma^4}{n^2}.$$

Thus, we have

$$\mathrm{var}\left(\overline{X}^2-\frac{S^2}{n(n-1)}\right)=\mathrm{var}(\overline{X}^2)+\mathrm{var}\left(\frac{S^2}{n(n-1)}\right)$$
$$=\frac{4\xi^2\sigma^2}{n}+\frac{2\sigma^4}{n^2}+\frac{\sigma^4}{n^2(n-1)^2}\mathrm{var}\left(\frac{S^2}{\sigma^2}\right)$$
$$=\frac{4\xi^2\sigma^2}{n}+\frac{2\sigma^4}{n^2}+\frac{2\sigma^4}{n^2(n-1)}.$$

Therefore, we obtain

$$R(\theta,\delta)=\frac{4\xi^2\sigma^2}{n}+\frac{2\sigma^4}{n^2}+\frac{2\sigma^4}{n^2(n-1)}.$$

c) **[+4]** Derive the UMVUE of ξ^4 if σ is known (with proof)

Ans:

If σ is known, we have

$$E(\overline{X}^4) = \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2}$$

and

$$E(\overline{X}^2) = \xi^2 + \frac{\sigma^2}{n} \Rightarrow E\left(\frac{6\sigma^2}{n}\overline{X}^2\right) = \frac{6\sigma^2\xi^2}{n} + \frac{6\sigma^4}{n^2}.$$

Thus,

$$\begin{aligned} E\left(\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}\right) &= E(\overline{X}^4) - \frac{6\sigma^2}{n}E(\overline{X}^2) + \frac{3\sigma^4}{n^2} \\ &= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \frac{6\sigma^2}{n}\xi^2 - \frac{6\sigma^4}{n^2} + \frac{3\sigma^4}{n^2} \\ &= \xi^4. \end{aligned}$$

Since $\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}$ is the UMVUE of ξ^4.

d) **[+2]** Derive the UMVUE of ξ^4 if σ is unknown (with proof)

<u>Ans:</u>

Since $\overline{X}$ and S^2 are independent, we have

$$E\left(\frac{S^2}{\sigma^2}\overline{X}^2\right) = E\left(\frac{S^2}{\sigma^2}\right)E(\overline{X}^2) = (n-1)\left(\xi^2 + \frac{\sigma^2}{n}\right)$$

$$\Rightarrow E\left(\frac{6S^2}{n(n-1)}\overline{X}^2\right) = \frac{6\sigma^2\xi^2}{n} + \frac{6\sigma^4}{n^2}$$

and

$$E\left(\frac{S^4}{\sigma^4}\right) = \text{var}\left(\frac{S^2}{\sigma^2}\right) + \left\{E\left(\frac{S^2}{\sigma^2}\right)\right\}^2 = 2(n-1) + (n-1)^2 = (n-1)(n+1) = n^2 - 1$$

$$\Rightarrow E\left(\frac{3S^4}{n^2(n^2-1)}\right) = \frac{3\sigma^4}{n^2}.$$

Therefore,

$$E\left(\overline{X}^4 - \frac{6S^2}{n(n-1)}\overline{X}^2 + \frac{3S^4}{n^2(n^2-1)}\right) = E(\overline{X}^4) - E\left(\frac{6S^2}{n(n-1)}\overline{X}^2\right) + E\left(\frac{3S^4}{n^2(n^2-1)}\right)$$

$$= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \frac{6\sigma^2\xi^2}{n} - \frac{6\sigma^4}{n^2} + \frac{3\sigma^4}{n^2}$$

$$= \xi^4.$$

Since $\overline{X}^4 - \frac{6S^2}{n(n-1)}\overline{X}^2 + \frac{3S^4}{n^2(n^2-1)}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^4 - \frac{6S^2}{n(n-1)}\overline{X}^2 + \frac{3S^4}{n^2(n^2-1)}$ is the UMVUE of ξ^4.

e) **[+6]** Under the constraint $\xi = \sigma$, derive the best estimator in a class of linear unbiased combinations of two unbiased estimators. Use the notation $c_n = \frac{1}{\sqrt{2}} \frac{\Gamma((n-1)/2)}{\Gamma(n/2)}$ to simplify the results.

Ans:

If $\xi = \sigma$, we have $X_1,..,X_n \overset{iid}{\sim} N(\xi, \xi^2)$. It is clear $E(\overline{X}) = \xi$.

Now, let

$$Y = \frac{S^2}{\xi^2} \sim \chi^2_{df=n-1} \Rightarrow S = \xi Y^{1/2}.$$

Then we have

$$\begin{aligned}
E(S) &= E(\xi Y^{1/2}) = \xi E(Y^{1/2}) = \xi \int_0^\infty y^{1/2} \frac{1}{2^{(n-1)/2}\Gamma((n-1)/2)} y^{\frac{n-1}{2}-1} e^{\frac{-y}{2}} dy \\
&= \frac{\xi 2^{n/2}\Gamma(n/2)}{2^{(n-1)/2}\Gamma((n-1)/2)} \int_0^\infty \frac{1}{2^{n/2}\Gamma(n/2)} y^{\frac{n}{2}-1} e^{\frac{-y}{2}} dy = \frac{\xi 2^{1/2}\Gamma(n/2)}{\Gamma((n-1)/2)} \\
&= c_n^{-1}\xi.
\end{aligned}$$

Therefore,

$$E(c_n S) = \xi.$$

Hence we obtain $\overline{X}$ and $c_n S$ are both unbiased estimator for ξ. Let $\delta = \alpha \overline{X} + (1-\alpha)c_n S$ is unbiased for ξ.

Then the risk

$$R(\theta, \delta) = E(\delta - \xi)^2 = \operatorname{var}(\delta) = \alpha^2 \operatorname{var}(\overline{X}) + (1-\alpha)^2 \operatorname{var}(c_n S).$$

$$\begin{aligned}
&\frac{\partial}{\partial \alpha} R(\theta, \delta) = 2\alpha \operatorname{var}(\overline{X}) - 2(1-\alpha)\operatorname{var}(c_n S) \equiv 0 \\
&\Rightarrow \alpha^* = \frac{\operatorname{var}(c_n S)}{\operatorname{var}(\overline{X}) + \operatorname{var}(c_n S)}.
\end{aligned}$$

Since

$$\operatorname{var}(\overline{X}) = \frac{\xi^2}{n}$$

and

$$\begin{aligned}
\operatorname{var}(c_n S) &= E(c_n^2 S^2) - E(c_n S)^2 = c_n^2 \xi^2 (n-1) - \xi^2 \\
&= \{c_n^2(n-1) - 1\}\xi^2.
\end{aligned}$$

Hence

$$\alpha^* = \frac{\{ c_n^2 (n-1) - 1 \} \xi^2}{\xi^2 / n + \{ c_n^2 (n-1) - 1 \} \xi^2} = \frac{n\{ c_n^2 (n-1) - 1 \}}{1 + n\{ c_n^2 (n-1) - 1 \}}.$$

Thus, the best estimator in the class of linear unbiased combination is

$$\alpha^* \overline{X} + (1 - \alpha^*) c_n S,$$

where

$$\alpha^* = \frac{n\{ c_n^2 (n-1) - 1 \}}{1 + n\{ c_n^2 (n-1) - 1 \}}.$$

Formula

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with σ^2 is known. The formula of $E(\overline{X}^k)$ is

$$E(\overline{X}^k) = \sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} E(Y^r),$$

where

$$Y \sim N(0, \sigma^2 / n)$$

and

$$E(Y^r) = \begin{cases} (r-1)(r-3)\cdots 3\cdot 1\cdot (\sigma^2/n)^{r/2} & \text{where } r \geq 2 \text{ is even} \\ 0 & \text{where } r \text{ is odd.} \end{cases}$$

Proof:

One can write $\overline{X} = Y + \xi$. By the binomial theorem, we have

$$\begin{aligned} E(\overline{X}^k) &= E\{(Y+\xi)^k\} \\ &= E\left\{ \sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} Y^r \right\} \\ &= \sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} E(Y^r). \end{aligned}$$

Then

$$E(Y^r) = \int_{-\infty}^{\infty} y^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{ \frac{-y^2}{2\sigma^2/n} \right\} dy.$$

Using the change of variable $y = \frac{\sigma}{\sqrt{n}} u$, then we can obtain

$$\begin{aligned} &\int_{-\infty}^{\infty} y^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{ \frac{-y^2}{2\sigma^2/n} \right\} dy \\ &= \int_{-\infty}^{\infty} \left(\frac{\sigma}{\sqrt{n}} u \right)^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{ \frac{-1}{2\sigma^2/n} \frac{\sigma^2}{n} u^2 \right\} \frac{\sigma}{\sqrt{n}} du \\ &= \left(\frac{\sigma}{\sqrt{n}} \right)^r \int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{ \frac{-u^2}{2} \right\} du. \end{aligned}$$

Then consider the integral

$$\int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du \ .$$

If r is odd, it is an integral of an odd function over a real line. Hence it is zero. If r is even, consider the change of variable $u = \sqrt{w}$, then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du &= 2\int_{0}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du \\
&= 2\int_{0}^{\infty} (\sqrt{2w})^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-(\sqrt{2w})^2}{2}\right\} \frac{1}{\sqrt{2w}} dw \\
&= \frac{2^{\frac{r}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} w^{\frac{r}{2}-\frac{1}{2}} e^{-w} dw = \frac{2^{\frac{r}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{\pi}} 2^{\frac{r}{2}} \left(\frac{r-1}{2}\right)\left(\frac{r-3}{2}\right) \cdots \frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi} \\
&= (r-1)(r-3)\cdots 3 \cdot 1.
\end{aligned}$$

Therefore, we obtain

$$E(Y^r) = \begin{cases} (r-1)(r-3)\cdots 3\cdot 1\cdot(\sigma^2/n)^{r/2} & \text{where } r \geq 2 \text{ is even} \\ 0 & \text{where } r \text{ is odd.} \end{cases}$$

Hence we have shown the formula of $E(\overline{X}^k)$.

Q2 [+22] We obtain independent observations $X_1,..,X_m \overset{iid}{\sim} N(\xi,\sigma^2)$ and $Y_1,..,Y_n \overset{iid}{\sim} N(\eta,\tau^2)$, where $\theta=(\xi,\sigma^2,\eta,\tau^2)$. Let $\overline{X}=\sum_{i=1}^m X_i/m$, $\overline{Y}=\sum_{i=1}^n Y_i/n$, $S_X^2=\sum_{i=1}^m (X_i-\overline{X})^2$, and $S_Y^2=\sum_{i=1}^n (Y_i-\overline{Y})^2$.

a) **[+3]** Derive the UMVUE of σ^r (with proof)

Ans:

$(\overline{X},\overline{Y},S_X^2,S_Y^2)$ is complete sufficient statistic for $\theta=(\xi,\sigma^2,\eta,\tau^2)$.

Now, let

$$W=\frac{S_X^2}{\sigma^2}\sim \chi^2_{df=m-1} \Rightarrow S=\sigma W^{1/2}.$$

Then we have

$$E(S_X^r)=E(\sigma^r W^{r/2})=\sigma^r E(W^{1/2})=\sigma^r\int_0^\infty w^{1/2}\frac{1}{2^{(m-1)/2}\Gamma((m-1)/2)}w^{\frac{m-1}{2}-1}e^{\frac{-w}{2}}dw$$

$$=\frac{\sigma^r 2^{(m+r-1)/2}\Gamma((m+r-1)/2)}{2^{(m-1)/2}\Gamma((m-1)/2)}\int_0^\infty \frac{1}{2^{(m+r-1)/2}\Gamma((m+r-1)/2)}w^{\frac{m+r-1}{2}-1}e^{\frac{-w}{2}}dw$$

$$=\frac{\sigma^r 2^{r/2}\Gamma((m+r-1)/2)}{\Gamma((m-1)/2)}.$$

Here we define

$$K_{m-1,r}=\frac{\Gamma((m-1)/2)}{2^{r/2}\Gamma((m+r-1)/2)}.$$

Therefore,

$$E(K_{m-1,r}S_X^r)=\sigma^r.$$

Since $K_{m-1,r}S_X^r$ is a function of complete sufficient statistic hence $K_{m-1,r}S_X^r$ is the UMVUE of σ^r.

b) **[+4]** Derive the UMVUE of τ^r / σ^r (with proof)

Ans:

Since $\tau^r / \sigma^r = \tau^r \sigma^{-r}$. Similarly, we have

$$E(K_{n-1,r} S_Y^r) = \tau^r.$$

This formula is also correct for negative in the constraint

$$m > -r + 1.$$

This is because the gamma function $\Gamma(\alpha)$, where $\alpha > 0$. Therefore,

$$E(K_{m-1,-r} S_X^{-r}) = \sigma^{-r}.$$

Also X and Y are independent. Thus,

$$E(K_{n-1,r} S_Y^r K_{m-1,-r} S_X^{-r}) = \tau^r \sigma^{-r}.$$

Since $K_{n-1,r} S_Y^r K_{m-1,-r} S_X^{-r}$ is a function of complete sufficient statistic hence $K_{n-1,r} S_Y^r K_{m-1,-r} S_X^{-r}$ is the UMVUE of τ^r / σ^r.

c) **[+2]** Derive the UMVUE of ξ / σ (with proof)

Ans:

We have

$$E(\overline{X}) = \xi$$

and

$$E(K_{m-1,-1} S_X^{-1}) = \sigma^{-1}.$$

Since $\overline{X}$ and S^2 are independent, we have

$$E(\overline{X} K_{m-1,-1} S_X^{-1}) = E(\overline{X}) E(K_{m-1,-1} S_X^{-1}) = \xi / \sigma.$$

Since $\overline{X} K_{m-1,-1} S_X^{-1}$ is a function of complete sufficient statistic hence $\overline{X} K_{m-1,-1} S_X^{-1}$ is the UMVUE of ξ / σ.

d) **[+4]** Derive the UMVUE of σ^r when $\tau^2 = \sigma^2$ (with proof)

Ans:

If $\tau^2 = \sigma^2$, then we have

$$X_1,..,X_m \overset{iid}{\sim} N(\xi, \sigma^2) \text{ and } Y_1,..,Y_n \overset{iid}{\sim} N(\eta, \sigma^2).$$

Then

$$\begin{aligned}
& p_\theta(x, y) \\
&= \frac{1}{(2\pi\sigma^2)^{m/2}(2\pi\sigma^2)^{n/2}} \exp\left\{ \frac{-1}{2\sigma^2}\sum_{i=1}^{m}(X_i - \xi)^2 \frac{-1}{2\sigma^2}\sum_{i=1}^{n}(Y_i - \eta)^2 \right\} \\
&= \frac{1}{(2\pi\sigma^2)^{(m+n)/2}} \exp\left\{ \frac{-1}{2\sigma^2}\left(\sum_{i=1}^{m} X_i^2 + \sum_{i=1}^{n} y_i^2 \right) + \frac{\xi}{\sigma^2}\sum_{i=1}^{m} X_i + \frac{\eta}{\sigma^2}\sum_{i=1}^{n} Y_i - \frac{\xi^2 + \eta^2}{2\sigma^2} \right\}.
\end{aligned}$$

Since $\Theta = \left\{ \left(\frac{-1}{2\sigma^2}, \frac{\xi}{\sigma^2}, \frac{\eta}{\sigma^2} \right); \xi \in R, \eta \in R \text{ and } \sigma^2 > 0 \right\}$ contains a 3-dimensional open rectangle hence it is full rank. Therefore,

$$\left(\sum_{i=1}^{m} X_i^2 + \sum_{i=1}^{n} y_i^2, \sum_{i=1}^{m} X_i, \sum_{i=1}^{n} Y_i \right)$$

is a complete sufficient statistic for $\theta = (\xi, \eta, \sigma^2)$.

$$(\overline{X}, \overline{Y}, S^2 = S_X^2 + S_Y^2)$$

is also a complete sufficient statistic for $\theta = (\xi, \eta, \sigma^2)$.

Now, let

$$W = \frac{S^2}{\sigma^2} \sim \chi^2_{df=m+n-2} \Rightarrow S = \sigma W^{1/2}.$$

Then we have

$$\begin{aligned}
& E(S^r) \\
&= E(\sigma^r W^{r/2}) = \sigma^r E(W^{1/2}) = \sigma^r \int_0^\infty w^{1/2} \frac{1}{2^{(m+n-2)/2}\Gamma((m+n-2)/2)} w^{\frac{m+n-2}{2}-1} e^{\frac{-w}{2}} dw \\
&= \frac{\sigma^r 2^{(m+n+r-2)/2}\Gamma((m+n+r-2)/2)}{2^{(m+n-2)/2}\Gamma((m+n-2)/2)} \\
&\qquad \times \int_0^\infty \frac{1}{2^{(m+n+r-2)/2}\Gamma((m+n+r-2)/2)} w^{\frac{m+n+r-2}{2}-1} e^{\frac{-w}{2}} dw \\
&= \frac{\sigma^r 2^{r/2}\Gamma((m+n+r-2)/2)}{\Gamma((m+n-2)/2)}.
\end{aligned}$$

Therefore,

$$E(K_{m+n-2,r} S^r) = \sigma^r.$$

Since $K_{m+n-2,r}S^r$ is a function of complete statistic hence $K_{m+n-2,r}S^r$ is the UMVUE of σ^r.

e) **[+5]** Derive the UMVUE of ξ when $\xi = \eta$ and $\sigma^2/\tau^2 = \gamma$ is known (with proof).

Ans:

If $\xi = \eta$ and $\sigma^2 = \gamma\tau^2$, then we have

$$X_1,..,X_m \overset{iid}{\sim} N(\xi, \gamma\tau^2) \text{ and } Y_1,..,Y_n \overset{iid}{\sim} N(\xi, \tau^2).$$

Then

$$\begin{aligned}
& p_\theta(x, y) \\
&= \frac{1}{(2\pi\gamma\tau^2)^{m/2}(2\pi\tau^2)^{n/2}} \exp\left\{ \frac{-1}{2\gamma\tau^2} \sum_{i=1}^{m} (X_i - \xi)^2 \frac{-1}{2\tau^2} \sum_{i=1}^{n} (Y_i - \xi)^2 \right\} \\
&= \frac{1}{\gamma^{m/2}(2\pi\tau^2)^{(m+n)/2}} \\
&\qquad \times \exp\left\{ \frac{-1}{2\gamma\tau^2} \left(\sum_{i=1}^{m} X_i^2 + \gamma \sum_{i=1}^{n} y_i^2 \right) + \frac{\xi}{\gamma\tau^2} \left(\sum_{i=1}^{m} X_i + \gamma \sum_{i=1}^{n} Y_i \right) - \frac{\xi^2(1+\gamma)}{2\gamma\tau^2} \right\}.
\end{aligned}$$

Since $\Theta = \left\{ \left(\frac{-1}{2\gamma\tau^2}, \frac{\xi}{\gamma\tau^2} \right); \xi \in R \text{ and } \tau^2 > 0 \right\}$ contains a 2-dimensional open rectangle hence it is full rank. Therefore,

$$\left(T_1 = \sum_{i=1}^{m} X_i^2 + \gamma \sum_{i=1}^{n} y_i^2, T_2 = \sum_{i=1}^{m} X_i + \gamma \sum_{i=1}^{n} Y_i \right)$$

is a complete sufficient statistic for $\theta = (\xi, \tau^2)$.

Thus,

$$E(m\overline{X} + \gamma n\overline{Y}) = (m + \gamma n)\xi \Rightarrow E\left(\frac{T_2}{m + \gamma n} \right) = \xi.$$

Since $\frac{T_2}{m + \gamma n}$ is a function of complete sufficient statistic hence $\frac{T_2}{m + \gamma n}$ is the UMVUE of ξ.

f) **[+4]** Find an unbiased estimator of ξ when $\xi = \eta$ (with proof).

Ans:

Since γ is unknown, we can estimate γ by

$$\hat{\gamma} = \frac{\hat{\sigma}^2}{\hat{\tau}^2},$$

where $\hat{\sigma}^2 = \frac{S_X^2}{m-1}$ and $\hat{\tau}^2 = \frac{S_Y^2}{n-1}$ are unbiased.

Then

$$E\left(\frac{m\overline{X} + \hat{\gamma}n\overline{Y}}{m+\hat{\gamma}n}\right) = E\left(\frac{m\overline{X}}{m+\hat{\gamma}n}\right) + E\left(\frac{\hat{\gamma}n\overline{Y}}{m+\hat{\gamma}n}\right).$$

Let

$$\hat{\alpha} = \frac{m}{m+\hat{\gamma}n} \quad \text{and} \quad 1-\hat{\alpha} = \frac{\hat{\gamma}n}{m+\hat{\gamma}n}.$$

Therefore,

$$\begin{aligned} E\left(\frac{m\overline{X} + \hat{\gamma}n\overline{Y}}{m+\hat{\gamma}n}\right) &= E(\hat{\alpha}\overline{X}) + E((1-\hat{\alpha})\overline{Y}) \\ &= E(\hat{\alpha})\xi + \{1 - E(\hat{\alpha})\}\xi \\ &= \xi. \end{aligned}$$

Hence $\frac{m\overline{X} + \hat{\gamma}n\overline{Y}}{m+\hat{\gamma}n}$ is an unbiased estimator of ξ.