+8

Quiz #1, Quality control, 2018 Fall [+ 8 points]

Name: Hsu, Jiing Huang

+3

Q1 [+3] Let X be the number of nonconforming items in n=100 samples and p = 0.01 be the fraction nonconforming. Let $\hat{p}$ be an unbiased estimator of p. Calculate

1) $P(\hat{p} \le 0.005)$
2) $P(\hat{p} \le 0.01)$
3) $P(\hat{p} \le 0.025)$

Answer up to 2 digits 0.xx using the approximation below:

```
round(0.99^(98),2)  = 0.37
round(0.99^(99),2)  = 0.37
round(0.99^(100),2) = 0.37
```

(1) Set $X \sim Bin(n=100, p=0.01)$

Let $\hat{P} = \frac{X}{n} \Rightarrow E(\hat{P}) = \frac{E[X]}{n} = \frac{nP}{n} = P$

hence, $\hat{P}$ is an unbiased estimator of P

(C1) $|X_i| < A$, For A is constant

(C2) $\sum_{i=1}^{n} \sigma_i^2 \to \infty$ as $n \to \infty$

(1) $P(\hat{P} \le 0.05) = P(\frac{X}{n} \le 0.005)$

$= P(X \le 0.5)$

$\Rightarrow P(X \le 0.5) = C_0^{100} (0.01)^0 (0.99)^{100} = 0.37$ #

(2) $P(\hat{P} \le 0.01) = P(X \le 1)$

$= C_0^{100} (0.01)^0 (0.99)^{100} + C_1^{100} (0.01)^1 (0.99)^{99}$

$= 0.37 + 0.37 = 0.74$ #

(3) $P(\hat{P} \le 0.025) = P(X \le 2.5)$

$= C_0^{100} (0.01)^0 (0.99)^{100} + C_1^{100} (0.01)^1 (0.99)^{99} + C_2^{100} (0.01)^2 (0.99)^{98} = 0.74 + 0.18$

$= 0.92$ #

+5

Q2. [+5] Let X_i, $i = 1, \ldots, n$ be independent random variables with $X_i = i$ w.p. p and $X_i = 0$ w.p. 1-p.

(1) Compute $\sum_{i=1}^{n} E[X_i]$

+1

$\sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} [iP + 0(1-P)] = \sum_{i=1}^{n} iP = P\sum_{i=1}^{n} i$

$= P\frac{n(n+1)}{2}$ #

(2) Compute $\sum_{i=1}^{n} Var[X_i]$

+1

$\sum_{i=1}^{n} Var(X_i) = \sum_{i=1}^{n} [E(X_i^2) - E(X_i)^2]$

$= \sum_{i=1}^{n} i^2 P(1-P) = P(1-P)\sum_{i=1}^{n} i^2 = P(1-P)\frac{n(n+1)(2n+1)}{6}$ #

(3) Compute $E(|X_i - E[X_i]|^3)$.

+1

$E(|X_i - E[X_i]|^3) = (i - iP)^3 P + (iP)^3 (1-P)$

$= i^3 (1-P)^3 P + i^3 P^3 (1-P)$

$= i^3 (1-P) P (P^2 + (1-P)^2)$ #

(4) Verity the Liapounov condition

+1

Let $S_n^{(k)} = \sum_{i=1}^{\infty} i^k \approx \frac{n^{k+1}}{k+1}$

$$\frac{\left[\sum_{i=1}^{n} E|X_i - E[X_i]|^3\right]^2}{\left(\sum_{i=1}^{n} Var[X_i]\right)^3} = \frac{[(1-P)P(P^2 + (1-P)^2)]^2 \left[\sum_{i=1}^{n} i^3\right]^2}{\left[P(1-P)\sum_{i=1}^{n} i^2\right]^3}$$

hence, $\sum_{i=1}^{n} i^3 \approx \frac{n^4}{4}$

$\sum_{i=1}^{n} i^2 \approx \frac{n^3}{3}$

$$\approx \frac{[(1-P)P(P^2 + (1-P)^2)]^2 \left[\frac{n^4}{4}\right]^2}{[P(1-P)]^3 \left[\frac{n^3}{3}\right]^3} = \text{const} \times \frac{n^8}{n^9}$$

$= \text{const} \times \frac{1}{n} \to 0$ as $n \to \infty$

(5) Apply the CLT.

+1

Under the Liapounov condition. $\frac{\left[\sum_{i=1}^{n} E|X_i - E[X_i]|^3\right]^2}{\left(\sum_{i=1}^{n} Var(X_i)\right)^3} \to 0$ as $n \to \infty$.

by the CLT

$$\frac{\sum_{i=1}^{n} X_i - \sum_{i=1}^{n} E(X_i)}{\sqrt{\sum_{i=1}^{n} Var(X_i)}} = \frac{\sum_{i=1}^{n} X_i - P\frac{n(n+1)}{2}}{\sqrt{P(1-P)\frac{n \times (n+1)(2n+1)}{6}}} \xrightarrow{d} N(0,1)$$ #