Quiz #1, Quality control, 2017 Spring [+14 points] +14

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Q1 [+3] Let X be the number of nonconforming items in n=100 samples and p = 0.01 be the fraction nonconforming. Let $\hat{p}$ be an unbiased estimator of p. Calculate

1) $P(\hat{p} \le 0.005)$
2) $P(\hat{p} \le 0.01)$
3) $P(\hat{p} \le 0.02)$

$X \sim Bin(100, 0.01)$

Answer up to 2 digits 0.xx using the approximation below:

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round(0.99^(98),2)  = 0.37
round(0.99^(99),2)  = 0.37
round(0.99^(100),2) = 0.37
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(1) $\hat{p} = \frac{X}{n} = \frac{X}{100}$

$P(\hat{p} \le 0.005) = P(\frac{X}{100} \le 0.005) = P(X \le [100 \times 0.005]) = P(X \le [0.5])$

$= \sum_{x=0}^{0} \binom{100}{x} (0.01)^x (0.99)^{100-x} = \binom{100}{0} (0.01)^0 (0.99)^{100} = (0.99)^{100} \approx 0.37$

(2) $P(\hat{p} \le 0.01) = P(X \le [100 \times 0.01]) = P(X \le 1) = \sum_{x=0}^{1} \binom{100}{x} (0.01)^x (0.99)^{100-x}$

$= \binom{100}{0} (0.99)^{100} + \binom{100}{1} (0.01) (0.99)^{99}$

$= (0.99)^{100} + 100 \times 0.01 \times (0.99)^{99}$

$\approx 0.37 + 0.37$

$= 0.74$

(3) $P(\hat{p} \le 0.02) = P(X \le [100 \times 0.02]) = P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$

$= \binom{100}{0} (0.01)^0 (0.99)^{100} + \binom{100}{1} (0.01)^1 (0.99)^{99} + \binom{100}{2} (0.01)^2 (0.99)^{98}$

$= (0.99)^{100} + 100 \times 0.01 \times (0.99)^{99} + \frac{100!}{2!\,98!} \times (0.01)^2 (0.99)^{98}$

$\approx 0.37 + 0.37 + \frac{100 \times 99}{2} \times (0.01)^2 \times 0.37$

$= 0.37 + 0.37 + 0.495 \times 0.37$

$= 0.92315$

≈ 0.92

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Q2. [+4] Let $X \sim Bin(n=22, p=0.5)$. Answer up to 2 digits 0.xx

1) Approximate $\Pr(X \le 10)$ by the normal approximation with continuity correction.

$$P(X \le 10) = P\left(\frac{X-np}{\sqrt{np(1-p)}} \le \frac{10-np+0.5}{\sqrt{np(1-p)}}\right) = P\left(Z \le \frac{10-22\times0.5+0.5}{\sqrt{22\times0.5\times0.5}}\right)$$

$$= P\left(Z \le \frac{10-11+0.5}{\sqrt{5.5}}\right) \approx P(Z \le -0.2132) = 1 - P(Z \le 0.2132)$$

$$= 1 - \Phi(0.2132) \approx 1 - \Phi(0.213) = 1 - 0.584 = 0.416 \approx 0.42.$$

2) Approximate $\Pr(X = 10)$ by the normal approximation with continuity correction.

$$P(X=10) = P(X \le 10) - P(X \le 9) = P\left(\frac{X-np}{\sqrt{np(1-p)}} \le \frac{10-np+0.5}{\sqrt{np(1-p)}}\right) - P\left(\frac{X-np}{\sqrt{np(1-p)}} \le \frac{9-np+0.5}{\sqrt{np(1-p)}}\right)$$

$$= P\left(Z \le \frac{10-11+0.5}{\sqrt{5.5}}\right) - P\left(Z \le \frac{9-11+0.5}{\sqrt{5.5}}\right) \approx P(Z \le -0.2132) - P(Z \le -0.6396)$$

$$\approx [1 - P(Z \le 0.2132)] - [1 - P(Z \le 0.6396)]$$

$$= P(Z \le 0.6396) - P(Z \le 0.2132)$$

$$\approx P(Z \le 0.64) - P(Z \le 0.213) = \Phi(0.64) - \Phi(0.213) = 0.739 - 0.584 = 0.155 \approx 0.16$$

3) Calculate the exact value $\Pr(X = 10)$ using

> choose(22,10) = 646646

$$P(X=10) = \binom{22}{10}(0.5)^{10}(0.5)^{22-10}$$

$$= \binom{22}{10}(0.5)^{22}$$

$$= 646649 \times (0.5)^{22}$$

$$\approx 0.15417$$

$$\approx 0.15$$

4) Does the criterion of the normal approximation hold for n and p ? Verify your answer.

check ① $np > 10$

② $0.1 \le p \le 0.9$

since ① $np = 22 \times 0.5 = 11 > 10$

② $0.1 \le 0.5 \le 0.9$

∴ the criterion of the normal approximation hold for n and p.

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Q3. [+3] Let $X_1,..,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$ and $d_2 = E[R/\sigma]$, where R is the range.

1) Show that the distribution of $W = R/\sigma$ does not depend on (μ,σ^2).

$R = \max_{i=1 \cdots n} X_i - \min_{i=1 \cdots n} X_i$

$W = \frac{R}{\sigma} = \frac{\max_i X_i - \mu + \mu - \min_i X_i}{\sigma} = \frac{\max_i X_i - \mu}{\sigma} - \frac{\min_i X_i - \mu}{\sigma}$

Let $Z_n = \frac{\max X_i - \mu}{\sigma} \sim N(0,1)$ $\Rightarrow W = Z_n - Z_1 \sim N(0,2)$ which does not depend on (μ, σ^2)

$Z_1 = \frac{\min X_i - \mu}{\sigma} \sim N(0,1)$

Should explain more clearly

2) Calculate the value d_2 when n=2 (with mathematical derivation)

$R = \max\{X_1, X_2\} - \min\{X_1, X_2\}$
$= |X_1 - X_2|$

$\frac{R}{\sigma} = \frac{|X_1 - X_2|}{\sigma} = \left| \frac{X_1-\mu}{\sigma} - \frac{X_2-\mu}{\sigma} \right| = |Z_1 - Z_2|$ where $Z_1 = \frac{X_1-\mu}{\sigma} \sim N(0,1)$, $Z_2 = \frac{X_2-\mu}{\sigma} \sim N(0,1)$

Let $Z = Z_1 - Z_2 \sim N(0,2) \Rightarrow \frac{Z}{\sqrt{2}} \sim N(0,1)$

$\therefore d_2 = E\left(\frac{R}{\sigma}\right) = E|Z_1 - Z_2| = E|Z| = \sqrt{2}\int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$

$= \frac{\sqrt{2}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |z| e^{-\frac{z^2}{2}} dz = \frac{\sqrt{2}}{\sqrt{2\pi}} \times 2 \int_0^{\infty} z e^{-\frac{z^2}{2}} dz = \frac{2}{\sqrt{\pi}} \int_0^{\infty} z e^{-\frac{z^2}{2}} dz$

$= -\frac{2}{\sqrt{\pi}} \left(e^{-\frac{z^2}{2}} \Big]_0^{\infty} \right) = -\frac{2}{\sqrt{\pi}} (0 - 1) = \frac{2}{\sqrt{\pi}} \approx \frac{2}{1.772454} \approx 1.1283$

3) We obtained sizes of chips (43, 44, 47, 46, 45). Find an estimate of the standard deviation by "range method". [up to 2 digits: 0.00]

$\hat{\sigma} = E\left(\frac{R}{d_2}\right) = E\left(\frac{47-43}{2.326}\right) = E\left(\frac{4}{2.326}\right) \approx 1.7196 \approx 1.72$

+4 **Q4.**[+4] Let $X_i, i=1,\dots,n \overset{iid}{\sim} N(\mu,\sigma^2)$. Consider testing $H_0:\mu=\mu_0$ vs. $H_1:\mu\neq\mu_0$ with level $\alpha=0.0027$. Let $\delta=\mu_1-\mu_0$ be the shift of mean.

1) The engineer reject H_0 when $|Z_0| \ge Z_{\alpha/2}$ where $Z_0 = \frac{\bar X-\mu_0}{\sigma/\sqrt n}$

i.e. $\left|\frac{\bar X-\mu_0}{\sigma/\sqrt n}\right| \ge Z_{0.0027/2} = Z_{0.00135} = 3$

2) Derive the producer's risk (Type I error)

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ is true}) = P(|Z_0|\ge Z_{\alpha/2} \mid \mu=\mu_0) = P\left(\left|\frac{\bar X-\mu_0}{\sigma/\sqrt n}\right|\ge 3 \mid \mu=\mu_0\right)$

$= P\left(\frac{\bar X-\mu_0}{\sigma/\sqrt n}\ge 3 \mid \mu=\mu_0\right) + P\left(\frac{\bar X-\mu_0}{\sigma/\sqrt n}\le -3 \mid \mu=\mu_0\right) = P\left(\frac{\bar X-\mu}{\sigma/\sqrt n}\ge 3\right)+P\left(\frac{\bar X-\mu}{\sigma/\sqrt n}\le -3\right) = 1-\Phi(3)+\Phi(-3) = 2-2\Phi(3) = 2-0.99865\times2$ $= 0.0027$

3) Under $H_1:\mu_1=\mu_0+\delta$, derive the consumer's risk (Type II error) in terms of δ, σ, and Φ.

$\beta = P(\text{accept } H_0 \mid H_0 \text{ is false}) = P(|Z_0|<Z_{\alpha/2} \mid \mu\neq\mu_0) = P\left(\left|\frac{\bar X-\mu_0}{\sigma/\sqrt n}\right|\le 3 \mid \mu\neq\mu_0\right) = P\left(-3\le \frac{\bar X-\mu}{\sigma/\sqrt n}+\frac{\mu-\mu_0}{\sigma/\sqrt n}\le 3 \mid \mu\neq\mu_0\right)$

$= P\left(-3-\frac{\delta\sqrt n}{\sigma}\le \frac{\bar X-\mu}{\sigma/\sqrt n}\le 3-\frac{\delta\sqrt n}{\sigma} \mid \mu\neq\mu_0\right) = \Phi\left(3-\frac{\delta\sqrt n}{\sigma}\right)-\Phi\left(-3-\frac{\delta\sqrt n}{\sigma}\right)$

$\left(= \Phi\left(Z_{0.00135}-\frac{\delta\sqrt n}{\sigma}\right)-\Phi\left(-3-\frac{\delta\sqrt n}{\sigma}\right)\right)$

4) Draw OC curve for $n=4$

(Plot must be clear and contain detailed information such that I understand it)

$d=\frac{\delta}{\sigma}$

$\beta(d) = \Phi(Z_{\alpha/2}-d\sqrt n)-\Phi(-Z_{\alpha/2}-d\sqrt n) = \Phi(3-2d)-\Phi(-3-2d)$

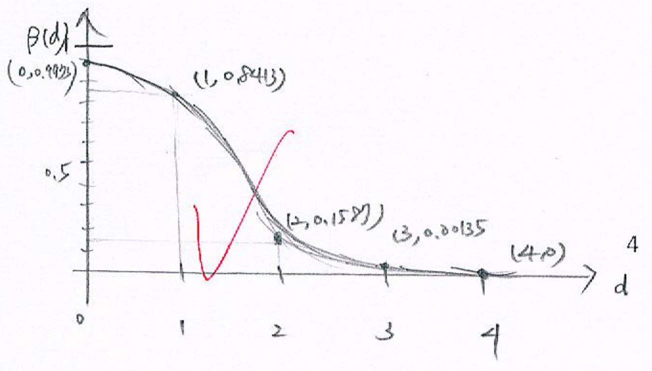
$\beta(0) = \Phi(3)-\Phi(-3) = \Phi(3)-(1-\Phi(3)) = 2\Phi(3)-1 = 0.9973$

$\beta(1) = \Phi(3-2)-\Phi(-3-2) = \Phi(1)-\Phi(-5) \approx \Phi(1)-0 = \Phi(1) \approx 0.8413$

$\beta(2) = \Phi(3-4)-\Phi(-3-4) = \Phi(-1)-\Phi(-7) \approx \Phi(-1)-0 = 1-\Phi(1) \approx 0.1587$

$\beta(3) = \Phi(3-6)-\Phi(-3-6) \approx \Phi(-3) = 1-\Phi(3) \approx 1-0.99865 = 0.00135$

$\beta(4) = \Phi(3-8)-\Phi(-3-8) \approx \Phi(-5)-\Phi(-11) \approx 0-0 = 0$



Tables

Table: the c.d.f. of N(0,1) $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$

z	0	0.213	0.5	0.64	1	1.5	1.96	2	2.5	3	3.5	4
P	0.5	0.584	0.6915	0.739	0.8413	0.9332	0.975	0.9772	0.9938	0.99865	0.99977	0.99997

Table: Conversion of range to standard deviation

n	2	3	4	5	6
d_2	1.128	1.683	2.059	2.326	2.534

Table: Square, square-root table

number	**square**	**square root**
3.141593	9.869604	1.772454
5	25	2.236
6	36	2.449
7	49	2.646
8	64	2.828
9	81	3.000
10	100	3.162
11	121	3.317
11.04	121.8816	3.32265
12	144	3.464