Quiz #2, Quality control 2015 Fall

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- **Not only answer but also derivations**
- **Numerical answer up to 3 digits 0.XXX**

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+5

1. Let $X \sim Bin(n=22, p=0.5)$ and $\hat{p}$ be an unbiased estimator of p.

+1 1) Approximate $\Pr(X=10)$ by the normal approximation.

+1 2) Calculate the exact value $\Pr(X=10)$

> choose(22,10) = 646646

1) $\mu = np = 11$, $\sigma = \sqrt{np(1-p)} = 2.345$

$Pr(X=10) \approx \Phi\left(\frac{10+0.5-11}{2.345}\right) - \Phi\left(\frac{10-0.5-11}{2.345}\right)$

$= \Phi(-0.213) - \Phi(-0.64)$

$= (1-\Phi(0.21)) - (1-\Phi(0.641))$

$= (1-0.58317) - (1-0.73891)$

$\doteq 0.156$ ✓

2) $Pr(X=10) = \binom{22}{10} 0.5^{10} 0.5^{12}$

$= 646646 \times 0.5^{10} 0.5^{12}$

$= 0.154$ ✓

+1 3) Approximate $\Pr(X \le 10)$ by the normal approximation.

+1 4) Approximate $\Pr(\hat{p} \le 0.4)$ by the normal approximation.

+1 5) Does the criterion of the normal approximation hold for n and p? Verify your answer.

3) $Pr(X \le 10) \approx \Phi\left(\frac{10+0.5-11}{2.345}\right)$

$= \Phi(-0.213)$

$= 1 - \Phi(0.21)$

$= 1 - 0.58317$

$\doteq 0.417$ ✓

4) $Pr(\hat{p} \le 0.4) = Pr\left(\frac{X}{n} \le 0.4\right)$

$= Pr(X \le 0.4 \times 22)$

$= Pr(X \le 8.8)$

$\approx \Phi\left(\frac{8.8-11}{2.345}\right)$

$= \Phi(-0.938)$

$= 1 - \Phi(0.94)$

$= 1 - 0.82639$

$\doteq 0.174$ ✓

5) ① by (3),

$Pr(X \le 10) = \sum_{x=0}^{10} \binom{22}{x} 0.5^x 0.5^{22-x}$

$= 0.5^{22} \left[\sum_{x=0}^{10} \binom{22}{x}\right]$

$= 0.5^{22}(1+22+231+1540+7315+26334+74613+170544+319770+497420+646646)$

$\doteq 0.416$

② by (4),

$Pr(\hat{p} \le 0.4) = Pr\left(\frac{X}{n} \le 0.4\right) = Pr(X \le 0.4 \times 22)$

$= Pr(X \le 8.8) = \sum_{x=0}^{8} \binom{22}{x} 0.5^x 0.5^{22-x}$

$= 0.5^{22} \times \left[\sum_{x=0}^{8} \binom{22}{x}\right]$

$\doteq 0.143$

Yes, the criterion of the normal approximation holds for n and p.

Criterion: $0.1 \le p \le 0.9$ and $np > 10$

$\rightarrow 0.1 \le 0.5 \le 0.9$ and $np = 11 > 10$ ∴ OK

+2

2. Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$ and $d_2 = E[R/\sigma]$, where R is the range.

1) Show that the distribution of $W = R/\sigma$ does not depend on (μ,σ^2).

2) Calculate the value d_2 when n=2 (with mathematical derivation)

1)
$$
\begin{aligned}
W &= \frac{R}{\sigma} \\
&= \frac{\max_i(X_i) - \min_i(X_i)}{\sigma} \\
&= \frac{\max_i(X_i) - \mu + \mu - \min_i(X_i)}{\sigma} \\
&= \frac{\max_i(X_i - \mu)}{\sigma} - \frac{\min_i(X_i - \mu)}{\sigma} \\
&= \max_i\left(\frac{X_i - \mu}{\sigma}\right) - \min_i\left(\frac{X_i - \mu}{\sigma}\right) \quad ✓ \\
&= \max_i Z_i - \min_i Z_i
\end{aligned}
$$

$\rightarrow W = \frac{R}{\sigma}$ does not depend on (μ, σ^2) #

2) Let $Z_1, Z_2 \sim N(0,1)$, $Z_1 - Z_2 \sim N(0,2)$, $\frac{Z_1 - Z_2}{\sqrt{2}} \sim N(0,1)$

$$
\begin{aligned}
d_2 &= E(W) \\
&= E(\max_i Z_i - \min_i Z_i) \\
&= \sqrt{2}\, E\left|\frac{Z_1 - Z_2}{\sqrt{2}}\right| \\
&= \sqrt{2}\, E|Z| \\
&= \sqrt{2} \int_{-\infty}^{\infty} |z| \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}\, dz \\
&= \frac{1}{\sqrt{\pi}} \int_0^{\infty} 2z\, e^{-\frac{z^2}{2}}\, dz \quad (\because z^2 dz = 2z dz) \\
&= \frac{2}{\sqrt{\pi}} e^{-\frac{z^2}{2}} \cdot (-2) \Big|_0^{\infty} \\
&= \frac{2}{\sqrt{\pi}} \\
&\fallingdotseq 1.128 \text{ #}
\end{aligned}
$$

✓