Final Exam, Quality Control, Fall 2014 [+~~20~~ 25 points]
Name: Lo, Yen-Wen

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Q1. [+10] Consider a 2×2 factorial experiment described as

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}, \quad i=1,2, \quad j=1,2, \quad k=1,2,\dots,n,$$

$$\varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2), \boxed{\sum_{i=1}^{2}\tau_i = \sum_{j=1}^{2}\beta_j = 0}_{①}, \boxed{\sum_{i=1}^{2}(\tau\beta)_{ij} = 0}_{②} \text{ for } \forall j \text{ and } \boxed{\sum_{j=1}^{2}(\tau\beta)_{ij} = 0}_{③} \text{ for } \forall i.$$

+5

1) [+5] Derive the estimator $\hat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta} = \{\mu, \tau_1, \tau_2, \beta_1, \beta_2, (\tau\beta)_{11}, (\tau\beta)_{12}, (\tau\beta)_{21}, (\tau\beta)_{22}\}$

$$y_{i..} = \sum_{j=1}^{2}\sum_{k=1}^{n}(\mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}) = 2n\mu + 2n\tau_i + n\underbrace{\sum_{j=1}^{2}\beta_j}_{0 \text{ by ①}} + n\underbrace{\sum_{j=1}^{2}(\tau\beta)_{ij}}_{0 \text{ by ③}} + \sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$\bar{y}_{i..} = \frac{1}{2n}y_{i..} = \mu + \tau_i + \frac{1}{2n}\sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$E(\bar{y}_{i..}) = \mu + \tau_i + \underbrace{E\left(\frac{1}{2n}\sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}\right)}_{0} = \mu + \tau_i \Rightarrow \boxed{\hat{\tau}_i = \bar{y}_{i..} - \underset{\text{by (2)}}{\hat{\mu}} = \bar{y}_{i..} - \bar{y}_{...}} \text{ --- (1)}$$

$$(\because \varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2) \Rightarrow mean = 0).$$

$$y_{...} = \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}(\mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}) = 4n\mu + 2n\underbrace{\sum_{i=1}^{2}\tau_i}_{0 \text{ by ①}} + 2n\underbrace{\sum_{j=1}^{2}\beta_j}_{0 \text{ by ①}} + n\sum_{i=1}^{2}\underbrace{\sum_{j=1}^{2}(\tau\beta)_{ij}}_{0 \text{ by ③}} + \sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$\bar{y}_{...} = \frac{1}{2\times 2\times n}y_{...} = \mu + \frac{1}{4n}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$E(\bar{y}_{...}) = \mu + \underbrace{E\left(\frac{1}{4n}\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}\right)}_{0} = \mu \Rightarrow \boxed{\hat{\mu} = \bar{y}_{...}} \longrightarrow (2)$$

$$(\because \varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2) \Rightarrow mean = 0)$$

$$y_{.j.} = \sum_{i=1}^{2}\sum_{k=1}^{n}(\mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}) = 2n\mu + n\underbrace{\sum_{i=1}^{2}\tau_i}_{0 \text{ by ①}} + 2n\beta_j + n\underbrace{\sum_{i=1}^{2}(\tau\beta)_{ij}}_{0 \text{ by ②}} + \sum_{i=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$\bar{y}_{.j.} = \frac{1}{2n}y_{.j.} = \mu + \beta_j + \frac{1}{2n}\sum_{i=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$E(\bar{y}_{.j.}) = \mu + \beta_j + \underbrace{E\left(\frac{1}{2n}\sum_{i=1}^{2}\sum_{k=1}^{n}\varepsilon_{ijk}\right)}_{0} = \mu + \beta_j \Rightarrow \boxed{\hat{\beta}_j = \bar{y}_{.j.} - \underset{\text{by (2)}}{\hat{\mu}} = \bar{y}_{.j.} - \bar{y}_{...}} \text{ --- (3)}$$

$$(\because \varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2) \Rightarrow mean = 0)$$

$$y_{ij.} = \sum_{k=1}^{n}(\mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}) = n\mu + n\tau_i + n\cdot\beta_j + n\cdot(\tau\beta)_{ij} + \sum_{k=1}^{n}\varepsilon_{ijk}$$

$$\bar{y}_{ij.} = \frac{1}{n}y_{ij.} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ijk}$$

$$E(\bar{y}_{ij.}) = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \underbrace{E\left(\frac{1}{n}\sum_{k=1}^{n}\varepsilon_{ijk}\right)}_{0} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} \Rightarrow \boxed{\widehat{(\tau\beta)}_{ij}} = \bar{y}_{ij.} - \hat{\mu} - \hat{\tau}_i - \hat{\beta}_j$$

$$(\because \varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2) \Rightarrow mean = 0) \quad (\text{by (1), (2), (3)}) = \bar{y}_{ij.} - \cancel{\bar{y}_{...}} - (\bar{y}_{i..} - \cancel{\bar{y}_{...}}) - (\bar{y}_{.j.} - \bar{y}_{...})$$

$$= \boxed{\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}}$$

Hence,

$$\hat{\boldsymbol{\theta}} = \begin{bmatrix} \hat{\mu} \\ \hat{\tau}_1 \\ \hat{\tau}_2 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \widehat{(\tau\beta)}_{11} \\ \widehat{(\tau\beta)}_{12} \\ \widehat{(\tau\beta)}_{21} \\ \widehat{(\tau\beta)}_{22} \end{bmatrix} = \begin{bmatrix} \bar{y}_{...} \\ \bar{y}_{1..} - \bar{y}_{...} \\ \bar{y}_{2..} - \bar{y}_{...} \\ \bar{y}_{.1.} - \bar{y}_{...} \\ \bar{y}_{.2.} - \bar{y}_{...} \\ \bar{y}_{11.} - \bar{y}_{1..} - \bar{y}_{.1.} + \bar{y}_{...} \\ \bar{y}_{12.} - \bar{y}_{1..} - \bar{y}_{.2.} + \bar{y}_{...} \\ \bar{y}_{21.} - \bar{y}_{2..} - \bar{y}_{.1.} - \bar{y}_{...} \\ \bar{y}_{22.} - \bar{y}_{2..} - \bar{y}_{.2.} - \bar{y}_{...} \end{bmatrix}$$

is the estimator of $\boldsymbol{\theta} = \{\mu, \tau_1, \tau_2, \beta_1, \beta_2, (\tau\beta)_{11}, (\tau\beta)_{12}, (\tau\beta)_{21}, (\tau\beta)_{22}\}$

#

+5 2) [+5] Show that $\hat{\boldsymbol\theta}$ is the LSE.

$$(\because\ (\mathbb{Y}'\cdot\mathbb{X}\cdot\boldsymbol\theta)' = \boldsymbol\theta'\cdot\mathbb{X}'\cdot\mathbb{Y})$$

$$S(\boldsymbol\theta) = \|\mathbb{Y}-\mathbb{X}\cdot\boldsymbol\theta\|^2 = (\mathbb{Y}-\mathbb{X}\cdot\boldsymbol\theta)'(\mathbb{Y}-\mathbb{X}\cdot\boldsymbol\theta) = \mathbb{Y}'\mathbb{Y} - \underbrace{\mathbb{Y}'\mathbb{X}\boldsymbol\theta} - \boldsymbol\theta'\mathbb{X}'\mathbb{Y} + \boldsymbol\theta'\mathbb{X}'\mathbb{X}\boldsymbol\theta = \underbrace{\mathbb{Y}'\mathbb{Y} - 2\boldsymbol\theta'\mathbb{X}'\mathbb{Y} + \boldsymbol\theta'\mathbb{X}'\mathbb{X}\boldsymbol\theta}$$

$$\frac{\partial}{\partial\boldsymbol\theta}S(\boldsymbol\theta)\Big|_{\hat{\boldsymbol\theta}} = -2\mathbb{X}'\mathbb{Y} + 2\mathbb{X}'\mathbb{X}\hat{\boldsymbol\theta} \overset{\text{set}}{=} 0 \Rightarrow \underbrace{\mathbb{X}'\mathbb{Y} = \mathbb{X}'\mathbb{X}\cdot\hat{\boldsymbol\theta}}$$

Check: $\mathbb{X}'\mathbb{Y} = \mathbb{X}'\mathbb{X}\cdot\hat{\boldsymbol\theta}$.

Constraints (satisfy):

$$\begin{cases}\sum_{i=1}^{2}\hat\tau_i = \sum_{i=1}^{2}(\bar y_{i..}-\bar y_{...}) = 2\cdot\bar y_{...} - 2\bar y_{...} = 0 \\ \sum_{j=1}^{2}\hat\beta_j = \sum_{j=1}^{2}(\bar y_{.j.}-\bar y_{...}) = 2\cdot\bar y_{...} - 2\bar y_{...} = 0 \\ \sum_{i=1}^{2}(\widehat{\tau\beta})_{ij} = \sum_{i=1}^{2}(\bar y_{ij.}-\bar y_{i..}-\bar y_{.j.}+\bar y_{...}) = \cancel{2\bar y_{.j.}} - \cancel{2\bar y_{...}} - \cancel{2\bar y_{.j.}} + \cancel{2\bar y_{...}} = 0, \quad \forall j=1,2 \\ \sum_{j=1}^{2}(\widehat{\tau\beta})_{ij} = \sum_{j=1}^{2}(\bar y_{ij.}-\bar y_{i..}-\bar y_{.j.}+\bar y_{...}) = \cancel{2\bar y_{i..}} - \cancel{2\bar y_{i..}} - \cancel{2\bar y_{...}} + \cancel{2\bar y_{...}} = 0, \quad \forall i=1,2\end{cases}$$

$$\Rightarrow \begin{cases}\hat\tau_1+\hat\tau_2 = 0 & \text{—① }\\ \hat\beta_1+\hat\beta_2 = 0 & \text{—②}\\ (\widehat{\tau\beta})_{11}+(\widehat{\tau\beta})_{21} = 0 & \text{—③}\\ (\widehat{\tau\beta})_{12}+(\widehat{\tau\beta})_{22} = 0 & \text{—④}\\ (\widehat{\tau\beta})_{11}+(\widehat{\tau\beta})_{12} = 0 & \text{—⑤}\\ (\widehat{\tau\beta})_{21}+(\widehat{\tau\beta})_{22} = 0 & \text{—③ − ⑤ + ④}\end{cases} \Rightarrow 5 \text{ constraints}$$

$$\therefore \underset{(\text{parameters})}{9} - \underset{(\text{constraints})}{5} = \underset{(\text{effective parameters})}{4}.$$

$\therefore$ Rank$(\mathbb{X}'\mathbb{X})$ = Rank$(\mathbb{X})$ = 4.

Let $\mathbb{Y} = \mathbb{X}\cdot\boldsymbol\theta + \boldsymbol\varepsilon$

$$\Rightarrow \begin{bmatrix} y_{111}\\ \vdots \\ y_{11n}\\ y_{121}\\ \vdots\\ y_{12n}\\ y_{211}\\ \vdots\\ y_{21n}\\ y_{221}\\ \vdots\\ y_{22n}\end{bmatrix} = \begin{bmatrix} 1&1&0&1&0&1&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&1&0&1&0&1&0&0&0\\ 1&1&0&0&1&0&1&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&1&0&0&1&0&1&0&0\\ 1&0&1&1&0&0&0&1&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&0&1&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&0&1&0&1&0&0&0&1 \end{bmatrix}\begin{bmatrix}\mu\\ \tau_1\\ \tau_2\\ \beta_1\\ \beta_2\\ (\tau\beta)_{11}\\ (\tau\beta)_{12}\\ (\tau\beta)_{21}\\ (\tau\beta)_{22}\end{bmatrix} + \begin{bmatrix}\varepsilon_{111}\\ \vdots\\ \varepsilon_{11n}\\ \varepsilon_{121}\\ \vdots\\ \varepsilon_{12n}\\ \varepsilon_{211}\\ \vdots\\ \varepsilon_{21n}\\ \varepsilon_{221}\\ \vdots\\ \varepsilon_{22n}\end{bmatrix}$$

$$\mathbb{X}'\mathbb{Y} = \begin{bmatrix} 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1\\ 1\cdots1 & 1\cdots1 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1\\ 1\cdots1 & 0\cdots0 & 1\cdots1 & 0\cdots0\\ 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1\\ 1\cdots1 & 0\cdots0 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 1\cdots1 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 \end{bmatrix}\begin{bmatrix} y_{111}\\ \vdots\\ y_{11n}\\ y_{121}\\ \vdots\\ y_{12n}\\ y_{211}\\ \vdots\\ y_{21n}\\ y_{221}\\ \vdots\\ y_{22n}\end{bmatrix} = n\cdot\begin{bmatrix}\bar y_{11.}+\bar y_{12.}+\bar y_{21.}+\bar y_{22.}\\ \bar y_{11.}+\bar y_{12.}\\ \bar y_{21.}+\bar y_{22.}\\ \bar y_{11.}+\bar y_{21.}\\ \bar y_{12.}+\bar y_{22.}\\ \bar y_{11.}\\ \bar y_{12.}\\ \bar y_{21.}\\ \bar y_{22.}\end{bmatrix} \text{ — ✪}$$

$$\mathbb{X}'\mathbb{X} = \begin{bmatrix} 1\cdots1 & 1\cdots1 & 1\cdots1 & 1\cdots1\\ 1\cdots1 & 1\cdots1 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 1\cdots1 & 1\cdots1\\ 1\cdots1 & 0\cdots0 & 1\cdots1 & 0\cdots0\\ 0\cdots0 & 1\cdots1 & 0\cdots0 & 1\cdots1\\ 1\cdots1 & 0\cdots0 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 1\cdots1 & 0\cdots0 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 1\cdots1 & 0\cdots0\\ 0\cdots0 & 0\cdots0 & 0\cdots0 & 1\cdots1 \end{bmatrix}\begin{bmatrix} 1&1&0&1&0&1&0&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&1&0&1&0&1&0&0&0\\ 1&1&0&0&1&0&1&0&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&1&0&0&1&0&1&0&0\\ 1&0&1&1&0&0&0&1&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&0&1&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1\\ \vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\ 1&0&1&0&1&0&0&0&1 \end{bmatrix} = \begin{bmatrix} 4n&2n&2n&2n&2n&n&n&n&n\\ 2n&2n&0&n&n&n&n&0&0\\ 2n&0&2n&n&n&0&0&n&n\\ 2n&n&n&2n&0&n&0&n&0\\ 2n&n&n&0&2n&0&n&0&n\\ n&n&0&n&0&n&0&0&0\\ n&n&0&0&n&0&n&0&0\\ n&0&n&n&0&0&0&n&0\\ n&0&n&0&n&0&0&0&n \end{bmatrix} = n\cdot\begin{bmatrix} 4&2&2&2&2&1&1&1&1\\ 2&2&0&1&1&1&1&0&0\\ 2&0&2&1&1&0&0&1&1\\ 2&1&1&2&0&1&0&1&0\\ 2&1&1&0&2&0&1&0&1\\ 1&1&0&1&0&1&0&0&0\\ 1&1&0&0&1&0&1&0&0\\ 1&0&1&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1 \end{bmatrix}$$

$$\mathbb{X}'\mathbb{X}\hat{\boldsymbol\theta} = n\cdot\begin{bmatrix} 4&2&2&2&2&1&1&1&1\\ 2&2&0&1&1&1&1&0&0\\ 2&0&2&1&1&0&0&1&1\\ 2&1&1&2&0&1&0&1&0\\ 2&1&1&0&2&0&1&0&1\\ 1&1&0&1&0&1&0&0&0\\ 1&1&0&0&1&0&1&0&0\\ 1&0&1&1&0&0&0&1&0\\ 1&0&1&0&1&0&0&0&1 \end{bmatrix}\begin{bmatrix}\bar y_{...}\\ \bar y_{1..}-\bar y_{...}\\ \bar y_{2..}-\bar y_{...}\\ \bar y_{.1.}-\bar y_{...}\\ \bar y_{.2.}-\bar y_{...}\\ \bar y_{11.}-\bar y_{1..}-\bar y_{.1.}+\bar y_{...}\\ \bar y_{12.}-\bar y_{1..}-\bar y_{.2.}+\bar y_{...}\\ \bar y_{21.}-\bar y_{2..}-\bar y_{.1.}+\bar y_{...}\\ \bar y_{22.}-\bar y_{2..}-\bar y_{.2.}+\bar y_{...}\end{bmatrix} = n\cdot\begin{bmatrix}\bar y_{11.}+\bar y_{12.}+\bar y_{21.}+\bar y_{22.}\\ \bar y_{11.}+\bar y_{12.}\\ \bar y_{21.}+\bar y_{22.}\\ \bar y_{11.}+\bar y_{21.}\\ \bar y_{12.}+\bar y_{22.}\\ \bar y_{11.}\\ \bar y_{12.}\\ \bar y_{21.}\\ \bar y_{22.}\end{bmatrix} \text{ — ✪✪}$$

Hence, By ✪ and ✪✪,

$$\mathbb{X}'\mathbb{X}\hat{\boldsymbol\theta} = \mathbb{X}'\cdot\mathbb{Y} \Rightarrow \hat{\boldsymbol\theta} = (\mathbb{X}'\mathbb{X})^{-1}\mathbb{X}'\cdot\mathbb{Y}.$$

$\Rightarrow \hat{\boldsymbol\theta}$ is the LSE. #

Q2 [+15]. An engineer designed a two-factor experiment with a =3 types and b =2 methods. The response is the adhesion force of the primer paint, which is described by

Model: $y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$, ~~$i = 1,2$~~, ~~$j = 1,2,3$~~, $k = 1,2,3$,

$i = 1,2,3;\ j = 1,2$

where $\varepsilon_{ijk} \overset{iid}{\sim} N(0,\sigma^2)$, $\sum_{i=1}^{a}\tau_i = \sum_{j=1}^{b}\beta_j = 0$, $\sum_{i=1}^{a}(\tau\beta)_{ij} = 0$ for $\forall j$ and $\sum_{j=1}^{b}(\tau\beta)_{ij} = 0$ for $\forall i$.

Data are obtained from the $n = 3$ experiments below:

Experiment 1

$j = 1, 2$

$i = 1, 2, 3$

	Dipping	Spraying
Tyep 1	4.0	5.4
Type 2	5.6	5.8
Type 3	3.8	5.5

Types

Experiment 2

	Dipping	Spraying
Tyep 1	4.5	4.9
Type 2	4.9	6.1
Type 3	3.7	5.0

Experiment 3

	Dipping	Spraying
Tyep 1	4.3	5.6
Type 2	5.4	6.3
Type 3	4.0	5.0

$n = 3$ Methods

	Dipping 1	Spraying 2	
Type 1	4.0, 4.5, 4.3 $\bar{y}_{11\cdot} = \frac{12.8}{3} \doteq 4.267$	5.4, 4.9, 5.6 $\bar{y}_{12\cdot} = \frac{15.9}{3} = 5.3$	$\bar{y}_{1\cdot\cdot} = \frac{28.7}{3\times2} \doteq 4.783$
Type 2	5.6, 4.9, 5.4 $\bar{y}_{21\cdot} = \frac{15.9}{3} \doteq 5.3$	5.8, 6.1, 6.3 $\bar{y}_{22\cdot} = \frac{18.2}{3} \doteq 6.067$	$\bar{y}_{2\cdot\cdot} = \frac{34.1}{3\times2} \doteq 5.683$
Type 3	3.8, 3.7, 4.0 $\bar{y}_{31\cdot} = \frac{11.5}{3} \doteq 3.833$	5.5, 5.0, 5.0 $\bar{y}_{32\cdot} = \frac{15.5}{3} \doteq 5.167$	$\bar{y}_{3\cdot\cdot} = \frac{27}{6} = 4.5$
	$\bar{y}_{\cdot1\cdot} = \frac{40.2}{3\times3} \doteq 4.467$	$\bar{y}_{\cdot2\cdot} = \frac{49.6}{3\times3} \doteq 5.511$	$\bar{y}_{\cdot\cdot\cdot} = \frac{89.8}{3\times2\times3} \doteq 4.989$

+3

1) [+3] Calculates the means: $\bar{y}_{ij\bullet}$, $\bar{y}_{i\bullet\bullet}$, $\bar{y}_{\bullet j\bullet}$, and $\bar{y}_{\bullet\bullet\bullet}$. [up to 3 disits; 0.000]

Which combination of the factors maximizes the adhesion force?

$(i^*, j^*) = \underset{(i,j)}{\arg\max}\ \bar{y}_{ij\cdot} = $ (Type 2 , Spraying) .

Hence, Type 2 and Spraying combination of the factors maximizes the adhesion force. ※

$i=1,2,3$; $j=1,2$; $k=1,2,3$

+3 2) [+3] Calculate the total sum squares: SS_T [up to 3 disits; 0.000]

$$SS_T = \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{3}(y_{ijk}-\bar{y}_{...})^2 \doteqdot [(4.0-4.989)^2+(4.5-4.989)^2+(4.3-4.989)^2+(5.4-4.989)^2+(4.9-4.989)^2+(5.6-4.989)^2$$
$$+(5.6-4.989)^2+(4.9-4.989)^2+(5.4-4.989)^2+(5.8-4.989)^2+(6.1-4.989)^2+(6.3-4.989)^2+(3.8-4.989)^2+(3.7-4.989)^2$$
$$+(4.0-4.989)^2+(5.5-4.989)^2+(5.0-4.989)^2+(5.0-4.989)^2] \doteqdot 10.718$$

※

+3 3) [+3] Calculate the sum squares: SS_A, SS_B, SS_{AB}, and SS_E [up to 3 disits; 0.000]

$$SS_A = \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{3}(\bar{y}_{i..}-\bar{y}_{...})^2 = 2\times3\cdot\sum_{i=1}^{3}(\bar{y}_{i..}-\bar{y}_{...})^2 = 6\times[(4.783-4.989)^2+(5.683-4.989)^2+(4.5-4.989)^2] \doteqdot 6\times0.763 = 4.578$$

$$SS_B = \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{3}(\bar{y}_{.j.}-\bar{y}_{...})^2 = 3\times3\cdot\sum_{j=1}^{2}(\bar{y}_{.j.}-\bar{y}_{...})^2 = 9\times[(4.467-4.989)^2+(5.511-4.989)^2] \doteqdot 9\times(0.272+0.272) = 9\times0.544 = 4.896$$

$$SS_{AB} = \sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{3}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2 = 3\cdot\left[\sum_{i=1}^{3}\sum_{j=1}^{2}(\bar{y}_{ij.}-\bar{y}_{i..}-\bar{y}_{.j.}+\bar{y}_{...})^2\right] = 3\cdot[(4.267-4.783-4.467+4.989)^2+(5.3-4.783-5.511+4.989)^2$$
$$+(5.3-5.683-4.467+4.989)^2+(6.067-5.683-5.511+4.989)^2+(3.833-4.5-4.467+4.989)^2+(5.167-4.5-5.511+4.989)^2]$$
$$\doteqdot 3\times0.080 = 0.240$$

$$SS_E = SS_T - SS_A - SS_B - SS_{AB} = 10.718-4.578-4.896-0.240 = 1.004$$

+3 4) [+3] Make the ANOVA table [up to 3 disits; 0.000]

	Source	(SS) Sum of square	(df) degrees of freedom	(MS) Mean square	F_0
(Types)	A	4.578	$df_A=(a-1)=3-1=2$	$\frac{SS_A}{df_A}=2.289$	$\frac{MS_A}{MS_E}=27.250$
(Methods)	B	4.896	$df_B=(b-1)=2-1=1$	$\frac{SS_B}{df_B}=4.896$	$\frac{MS_B}{MS_E}\doteqdot58.286$
	AB	0.240	$df_{AB}=(a-1)(b-1)=(3-1)(2-1)=2$	$\frac{SS_{AB}}{df_{AB}}=0.12$	$\frac{MS_{AB}}{MS_E}\doteqdot1.429$
	E	1.004	$df_E=ab(n-1)=3\cdot2\cdot(3-1)=12$	$\frac{SS_E}{df_E}=0.084$	
	Total	10.718	$df_T=abn-1=3\cdot2\cdot3-1=17$		

+3 5) [+3] Define the 3 hypotheses in the model. Then, test the hypotheses at 5% level. $\alpha=0.05$

① Testing interaction :

$H_0: (\tau\beta)_{ij}=0,\ \forall i=1,2,3;\ \forall j=1,2$.

$F_0 = \frac{MS_{AB}}{MS_E} = \frac{SS_{AB}/2}{SS_E/12} \doteqdot 1.429 \sim F_{2,12}$, $F_{\alpha=0.05,2,12}=3.89$.

$\Rightarrow F_0 \doteqdot 1.429 < F_{\alpha=0.05,2,12}=3.89 \Rightarrow$ With level $\alpha=0.05$, not reject $H_0: (\tau\beta)_{ij}=0,\ \forall i=1,2,3;\ \forall j=1,2$.

∴ There is no interaction between Type and Method. ※

② Testing main effect :

(1) $H_0: \tau_i=0,\ \forall i=1,2,3$. $F_0=\frac{MS_A}{MS_E}=\frac{SS_A/2}{SS_E/12}\doteqdot 27.25 \sim F_{2,12}$, $F_{\alpha=0.05,2,12}=3.89$.

$\Rightarrow F_0 = 27.25 > F_{\alpha=0.05,2,12}=3.89 \Rightarrow$ With level $\alpha=0.05$, reject $H_0: \tau_i=0,\ \forall i=1,2,3$

∴ Type (Factor A) has a significant effect on the adhesion force. ※

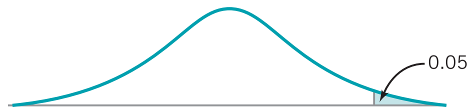
(2) $H_0: \beta_j=0,\ \forall j=1,2$. $F_0=\frac{MS_B}{MS_E}=\frac{SS_B/1}{SS_E/12}\doteqdot 58.286 \sim F_{1,12}$, $F_{\alpha=0.05,1,12}=4.75$.

$\Rightarrow F_0 \doteqdot 58.286 > F_{\alpha=0.05,1,12}=4.75 \Rightarrow$ With level $\alpha=0.05$, reject $H_0: \beta_j=0,\ \forall j=1,2$.

∴ Method (Factor B) has a significant effect on the adhesion force. ※

■ **APPENDIX V**

(Continued)



$F_{0.05,\, v_1,\, v_2}$

Degrees of freedom for the numerator (v_1)

v_2 \ v_1	1	2	3	4	5	6	7	8	9	10	12	15	20	24	30	40	60	120	∞
1	161.4	199.5	215.7	224.6	230.2	234.0	236.8	238.9	240.5	241.9	243.9	245.9	248.0	249.1	250.1	251.1	252.2	253.3	254.3
2	18.51	19.00	19.16	19.25	19.30	19.33	19.35	19.37	19.38	19.40	19.41	19.43	19.45	19.45	19.46	19.47	19.48	19.49	19.50
3	10.13	9.55	9.28	9.12	9.01	8.94	8.89	8.85	8.81	8.79	8.74	8.70	8.66	8.64	8.62	8.59	8.57	8.55	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.09	6.04	6.00	5.96	5.91	5.86	5.80	5.77	5.75	5.72	5.69	5.66	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.88	4.82	4.77	4.74	4.68	4.62	4.56	4.53	4.50	4.46	4.43	4.40	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.21	4.15	4.10	4.06	4.00	3.94	3.87	3.84	3.81	3.77	3.74	3.70	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.79	3.73	3.68	3.64	3.57	3.51	3.44	3.41	3.38	3.34	3.30	3.27	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.50	3.44	3.39	3.35	3.28	3.22	3.15	3.12	3.08	3.04	3.01	2.97	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.29	3.23	3.18	3.14	3.07	3.01	2.94	2.90	2.86	2.83	2.79	2.75	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.14	3.07	3.02	2.98	2.91	2.85	2.77	2.74	2.70	2.66	2.62	2.58	2.54
11	4.84	3.98	3.59	3.36	3.20	3.09	3.01	2.95	2.90	2.85	2.79	2.72	2.65	2.61	2.57	2.53	2.49	2.45	2.40
12	4.75	3.89	3.49	3.26	3.11	3.00	2.91	2.85	2.80	2.75	2.69	2.62	2.54	2.51	2.47	2.43	2.38	2.34	2.30
13	4.67	3.81	3.41	3.18	3.03	2.92	2.83	2.77	2.71	2.67	2.60	2.53	2.46	2.42	2.38	2.34	2.30	2.25	2.21
14	4.60	3.74	3.34	3.11	2.96	2.85	2.76	2.70	2.65	2.60	2.53	2.46	2.39	2.35	2.31	2.27	2.22	2.18	2.13
15	4.54	3.68	3.29	3.06	2.90	2.79	2.71	2.64	2.59	2.54	2.48	2.40	2.33	2.29	2.25	2.20	2.16	2.11	2.07
16	4.49	3.63	3.24	3.01	2.85	2.74	2.66	2.59	2.54	2.49	2.42	2.35	2.28	2.24	2.19	2.15	2.11	2.06	2.01
17	4.45	3.59	3.20	2.96	2.81	2.70	2.61	2.55	2.49	2.45	2.38	2.31	2.23	2.19	2.15	2.10	2.06	2.01	1.96
18	4.41	3.55	3.16	2.93	2.77	2.66	2.58	2.51	2.46	2.41	2.34	2.27	2.19	2.15	2.11	2.06	2.02	1.97	1.92
19	4.38	3.52	3.13	2.90	2.74	2.63	2.54	2.48	2.42	2.38	2.31	2.23	2.16	2.11	2.07	2.03	1.98	1.93	1.88
20	4.35	3.49	3.10	2.87	2.71	2.60	2.51	2.45	2.39	2.35	2.28	2.20	2.12	2.08	2.04	1.99	1.95	1.90	1.84
21	4.32	3.47	3.07	2.84	2.68	2.57	2.49	2.42	2.37	2.32	2.25	2.18	2.10	2.05	2.01	1.96	1.92	1.87	1.81
22	4.30	3.44	3.05	2.82	2.66	2.55	2.46	2.40	2.34	2.30	2.23	2.15	2.07	2.03	1.98	1.94	1.89	1.84	1.78
23	4.28	3.42	3.03	2.80	2.64	2.53	2.44	2.37	2.32	2.27	2.20	2.13	2.05	2.01	1.96	1.91	1.86	1.81	1.76
24	4.26	3.40	3.01	2.78	2.62	2.51	2.42	2.36	2.30	2.25	2.18	2.11	2.03	1.98	1.94	1.89	1.84	1.79	1.73
25	4.24	3.39	2.99	2.76	2.60	2.49	2.40	2.34	2.28	2.24	2.16	2.09	2.01	1.96	1.92	1.87	1.82	1.77	1.71
26	4.23	3.37	2.98	2.74	2.59	2.47	2.39	2.32	2.27	2.22	2.15	2.07	1.99	1.95	1.90	1.85	1.80	1.75	1.69
27	4.21	3.35	2.96	2.73	2.57	2.46	2.37	2.31	2.25	2.20	2.13	2.06	1.97	1.93	1.88	1.84	1.79	1.73	1.67
28	4.20	3.34	2.95	2.71	2.56	2.45	2.36	2.29	2.24	2.19	2.12	2.04	1.96	1.91	1.87	1.82	1.77	1.71	1.65
29	4.18	3.33	2.93	2.70	2.55	2.43	2.35	2.28	2.22	2.18	2.10	2.03	1.94	1.90	1.85	1.81	1.75	1.70	1.64
30	4.17	3.32	2.92	2.69	2.53	2.42	2.33	2.27	2.21	2.16	2.09	2.01	1.93	1.89	1.84	1.79	1.74	1.68	1.62
40	4.08	3.23	2.84	2.61	2.45	2.34	2.25	2.18	2.12	2.08	2.00	1.92	1.84	1.79	1.74	1.69	1.64	1.58	1.51
60	4.00	3.15	2.76	2.53	2.37	2.25	2.17	2.10	2.04	1.99	1.92	1.84	1.75	1.70	1.65	1.59	1.53	1.47	1.39
120	3.92	3.07	2.68	2.45	2.29	2.17	2.09	2.02	1.96	1.91	1.83	1.75	1.66	1.61	1.55	1.55	1.43	1.35	1.25
∞	3.84	3.00	2.60	2.37	2.21	2.10	2.01	1.94	1.88	1.83	1.75	1.67	1.57	1.52	1.46	1.39	1.32	1.22	1.00

Degrees of freedom for the denominator (v_2)

Note: $F_{0.95,\, v_1,\, v_2} = 1/F_{0.05,\, v_2,\, v_1}$.