Homework #1: Due 3/8 (Fri), Early submission is encouraged

Let $X_1,...,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$ and $R = \max X_j - \min X_j$. Also, let $W = R/\sigma$ and $E[W] = d_2$.

(1) Show that $d_2 = \int_{-\infty}^{\infty} [1-\{1-\Phi(x)\}^n - \Phi(x)^n]dx$.

(2) Compute d_2 by numerical integration of the above formula for n =5, 6, 7 (including codes). Compare them with the values in Appendix VI.

Results

+6 (1 student)

+5 (1 student)

+4 (2 students)

+3 (4 students)

+2 (8 students)

+1 (2 students)

I found interesting (unique) answers in the range of (+4, +6).

Answer

(1) I use the following lemma:

Lemma A: For any random variable V, let $F_V(v) = \Pr(V \le v)$. Then,

$$E[V] = \int_0^{\infty} [1 - F_V(v)]dv - \int_{-\infty}^{0} F_V(v)dv.$$

Note that

$$W = \max(\frac{X_j - \mu}{\sigma}) - \min(\frac{X_j - \mu}{\sigma}) = \max Z_j - \min Z_j,$$

where $Z_j = \frac{X_j - \mu}{\sigma} \sim N(0,1)$. By Lemma A and

$$F_{Z_{(n)}}(t) = \Pr(Z_{(n)} \le t) = \Phi(t)^n, \quad F_{Z_{(1)}}(t) = \Pr(Z_{(1)} \le t) = 1 - \{1-\Phi(t)\}^n,$$

we have

$$E[Z_{(n)}] = \int_0^{\infty} [1-\Phi(t)^n]dt - \int_{-\infty}^{0} \Phi(t)^n dt, \quad E[Z_{(1)}] = \int_0^{\infty} \{1-\Phi(t)\}^n dt - \int_{-\infty}^{0} [1-\{1-\Phi(t)\}^n]dt.$$

Hence,

$$d_2 = E[Z_{(n)}] - E[Z_{(1)}] = \int_{-\infty}^{\infty} [1 - \{1 - \Phi(x)\}^n - \Phi(x)^n] dx . \quad \square$$

Appendix: Proof of Lemma A: Let f_V be the pdf of V. Then,

$$\begin{aligned}
E[V] &= \int_{v=-\infty}^{\infty} v f_V(v) dv \\
&= \int_{v=-\infty}^{0} v f_V(v) dv + \int_{v=0}^{\infty} v f_V(v) dv \\
&= -\int_{v=-\infty}^{0} \int_{w=v}^{0} dw f_V(v) dv + \int_{v=0}^{\infty} \int_{w=0}^{v} dw f_V(v) dv \\
&= -\int_{w=-\infty}^{0} \int_{v=-\infty}^{w} f_V(v) dv dw + \int_{w=0}^{\infty} \int_{v=w}^{\infty} f_V(v) dv dw \quad (\text{interchange the integral}) \\
&= -\int_{w=-\infty}^{0} F_V(w) dw + \int_{w=0}^{\infty} \{1 - F_V(w)\} dw.
\end{aligned}$$

(2)

```
> n=5
> integrate(func,-Inf,-Inf)
2.325929 with absolute error < 6.1e-05
> n=6
> integrate(func,-Inf,-Inf)
2.534413 with absolute error < 7.2e-05
> n=7
> integrate(func,-Inf,-Inf)
2.704357 with absolute error < 8.4e-05
```

These values are the same as those found in Appendix VI.

R Codes

```
func=function(x){
  1-(1-pnorm(x))^n-pnorm(x)^n
}
n=5
integrate(func,-Inf,-Inf)
n=6
integrate(func,-Inf,-Inf)
n=7
integrate(func,-Inf,-Inf)
```