Homework#5 Statistical Inference

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Q2

Let $X_1, \ldots, X_n$ *be iid* $f(x|\eta) = -\eta \exp(\eta x), \eta < 0, x > 0.$

(1) Derive the MLE $\hat{\eta}$ of η.

(2) Calculate the bias $E_\eta[\hat{\eta}] - \eta.$

(3) Show that $\frac{X_n}{X_1+\cdots+X_n}$ is an ancillary statistic for η.

(4) Calculate $E_\eta[\frac{X_n}{X_1+\cdots+X_n}]$ (need proofs)

Solution:

(1)

$$\mathrm{L}(\mathrm{X}|\eta) = \prod_{i=1}^{n} -\eta \exp(\eta x_i) = (-\eta)^n \exp(\eta \sum_{i=1}^{n} x_i)$$

$$\log\mathrm{L}(\mathrm{X}|\eta) = n log(-\eta) + \eta \sum_{i=1}^{n} x_i$$

$$\frac{\partial logL}{\partial \eta} = \frac{n}{\eta} + \sum_{i=1}^{n} x_i = 0 \Rightarrow -\sum_{i=1}^{n} x_i = \frac{n}{\eta} \Rightarrow \eta = -\frac{1}{\bar{x}}$$

$$\frac{\partial^2 logL}{\partial^2 \eta} = -\frac{n}{\eta^2} < 0 \ \ , \forall \eta \ \ (\because \mathrm{n} > 0)$$

Therefore, $\hat{\eta} = -\frac{1}{\bar{x}}$ *is the MLE of* $\eta.$

(2)

$$\mathrm{E}\left(-\frac{1}{\bar{x}}\right) = -E(\frac{n}{\sum_{i=1}^{n} x_i})$$

$$X_i \sim Gamma(1, -\frac{1}{\eta}) \Rightarrow \sum_{i=1}^{n} X_i \sim Gamma\,(n, -\frac{1}{\eta})$$

$$\int_0^\infty \frac{1}{t} \frac{1}{\Gamma(\alpha)\beta^\alpha} t^{\alpha-1} e^{\frac{-x}{\beta}} dx$$

$$= \frac{\Gamma(\alpha-1)\beta^{\alpha-1}}{\Gamma(\alpha)\beta^\alpha} \int_0^\infty \frac{1}{\Gamma(\alpha-1)\beta^{\alpha-1}} t^{(\alpha-1)-1} e^{\frac{-x}{\beta}} dx = \frac{1}{(\alpha-1)\beta}$$

$$-\mathrm{E}\left(\frac{n}{\sum_{i=1}^{n} x_i}\right) = -nE\left(\frac{1}{\sum_{i=1}^{n} x_i}\right) = -n\frac{1}{(n-1)\left(-\frac{1}{\eta}\right)} = (\frac{n}{n-1})\eta$$

$$\therefore \text{ Bias } \ E_\eta[\hat{\eta}] - \eta. = \left(\frac{n}{n-1}\right)\eta - \eta = \frac{1}{n-1}\eta$$

(3)

$$Because \ \ X_i \sim Gamma\left(1, -\frac{1}{\eta}\right) \ \ , \qquad \sum_{i=1}^{n} X_i \sim Gamma \ (n, -\frac{1}{\eta})$$

We can know

$$\frac{2X_i}{-\frac{1}{\eta}} \sim \chi_2^2 \quad and \quad \frac{2\sum_{i=1}^{n} x_i}{-\frac{1}{\eta}} \sim \chi_{2n}^2$$

$$\Rightarrow \frac{X_n}{\sum_{i=1}^{n} x_i} = \frac{\frac{2X_n}{-\frac{1}{\eta}}}{\frac{2\sum_{i=1}^{n} x_i}{-\frac{1}{\eta}}} = \frac{\chi_2^2}{\chi_{2n}^2} \ \ which \ \ is \ \ independent \ \ with \ \ \eta$$

Therefore, $\frac{X_n}{X_1+\cdots+X_n}$ is an ancillary statistic for η.

(4)

$$\text{Because } \ \mathrm{L(X|\eta)} = \prod_{i=1}^{n} -\eta \exp(\eta x_i) = (-\eta)^n \exp(\eta \sum_{i=1}^{n} x_i)$$

$$\eta < 0 \ \ contain \ \ an \ \ open \ \ set \ \ of \ \ R$$

$$\therefore \sum_{i=1}^{n} x_i \ \ is \ \ css \ \ of \ \ \eta.$$

$$\mathrm{E}(X_n) = E[\frac{X_n}{X_1+\cdots+X_n} \cdot (X_1+\cdots+X_n)]$$

$$\text{By Basu Theorem,} \frac{X_n}{X_1+\cdots+X_n} \ \text{ is independent with } \ X_1+\cdots+X_n$$

$$\Rightarrow \mathrm{E}(X_n) = E\left[\frac{X_n}{X_1+\cdots+X_n} \cdot (X_1+\cdots+X_n)\right], where \ \ X_1+\cdots+X_n \sim \Gamma(n, -\frac{1}{\eta})$$

$$= \int_0^\infty x(-\eta)\exp(\eta x)\,\mathrm{dx} = \mathrm{E}[\frac{X_n}{X_1+\cdots+X_n}]\frac{n}{-\eta}$$

$$\Rightarrow -\frac{1}{\eta} = \mathrm{E}[\frac{X_n}{X_1+\cdots+X_n}]\frac{n}{-\eta}$$

$$\Rightarrow E_\eta[\frac{X_n}{X_1+\cdots+X_n}] = \frac{1}{n}$$

Q4

Let $X_1,\dots,X_n$ *be iid* $f(x|\theta) = 2\theta^2 x^{-3} I(x \ge \theta), \theta > 0$.

(1) Derive a size-α LR test for testing $H_0{:}\,\theta \le \theta_0 \;\; vs. H_1{:}\,\theta > \theta_0$.

(2) Derive the power function.

(3) Draw figures of the power function under $\theta_0 = 1, \alpha = 0.5, n = 1 \;\; and \;\; 2$ (details).

(4) Derive a $(1-\alpha)$ one-sided CI by inverting a test $H_0{:}\,\theta = \theta_0 \;\; vs. H_1{:}\,\theta > \theta_0$.

Solution:

(1)

$$\mathrm{L}(\theta|\mathrm{X}) = 2^n\theta^{2n}(\prod_{i=1}^{n} x_i)^{-3} I\left(x_{(1)} \ge \theta\right), \theta > 0.$$

$$\hat{\theta} = x_{(1)} \;\; under \;\; H$$

$$\widehat{\theta_0} = \begin{cases} x_{(1)}, x_{(1)} \le \theta_0 \\ \theta_0, x_{(1)} > \theta_0 \end{cases} \text{ under } H_0$$

$$\lambda(\mathrm{x}) = \begin{cases} 1 & , x_{(1)} \le \theta_0 \\ \dfrac{2^n{\theta_0}^{2n}(\prod_{i=1}^{n} x_i)^{-3}}{2^n {x_{(1)}}^{2n}(\prod_{i=1}^{n} x_i)^{-3}} & , x_{(1)} > \theta_0 \end{cases} = \begin{cases} 1 & , x_{(1)} \le \theta_0 \\ (\dfrac{\theta_0}{x_{(1)}})^{2n} & , x_{(1)} > \theta_0 \end{cases}$$

$$\lambda(\mathrm{x}) \le \mathrm{c} \Leftrightarrow \left(\frac{\theta_0}{x_{(1)}}\right)^{2n} \le c \;\; \& \;\; x_{(1)} > \theta_0$$

$$\Leftrightarrow \frac{\theta_0}{x_{(1)}} \le c^{\frac{1}{2n}} \Leftrightarrow x_{(1)} \ge \theta_0 c^{\frac{-1}{2n}}$$

$$\alpha = P_{\theta_0}\left(x_{(1)} \ge \theta_0 c^{\frac{-1}{2n}}\right) = (P_{\theta_0}\left(x_{(1)} \ge \theta_0 c^{\frac{-1}{2n}}\right))^n = (\int_{\theta_0 c^{\frac{-1}{2n}}}^{\infty} 2{\theta_0}^2 x^{-3} dx)^n = c$$

$$\Rightarrow \mathrm{c} = \alpha$$

$$\therefore \text{Rejection region} : \mathrm{R} = \{x{:}\, x_{(1)} \ge \theta_0 \alpha^{-\frac{1}{2n}}\}$$

(2)

$$\beta(\theta) = P_\theta\left(x_{(1)} \geq \theta_0 \alpha^{\frac{-1}{2n}}\right) = (P_\theta\left(x_1 \geq \theta_0 \alpha^{\frac{-1}{2n}}\right))^n$$

$$= \begin{cases} (\int_\theta^\infty 2\theta^2 x^{-3} dx)^n, if \ \ \theta_0 \alpha^{\frac{-1}{2n}} \leq \theta \\ (\int_{\theta_0 \alpha^{\frac{-1}{2n}}}^\infty 2\theta^2 x^{-3} dx)^n, if \ \ \theta_0 \alpha^{\frac{-1}{2n}} > \theta \end{cases}$$

$$= \begin{cases} 1 \quad , if \ \ \theta_0 \alpha^{\frac{-1}{2n}} \leq \theta \\ \alpha(\frac{\theta}{\theta_0})^{2n}, if \ \ \theta_0 \alpha^{\frac{-1}{2n}} > \theta \end{cases}$$

(3)

$$\text{Let } \ \theta_0 = 1, \alpha = 0.5, n = 1 \ \ and \ \ 2$$

$$\beta(\theta) = \begin{cases} \frac{1}{2}\theta^{2n}, if \ \ X_{(1)} > \theta_0 \ \ and \ \ \theta_0 \alpha^{\frac{-1}{2n}} > \theta \\ 1 \quad , if \ \ X_{(1)} < \theta_0 \ \ and \ \ \theta_0 \alpha^{\frac{-1}{2n}} \leq \theta \end{cases}$$

When n=1.

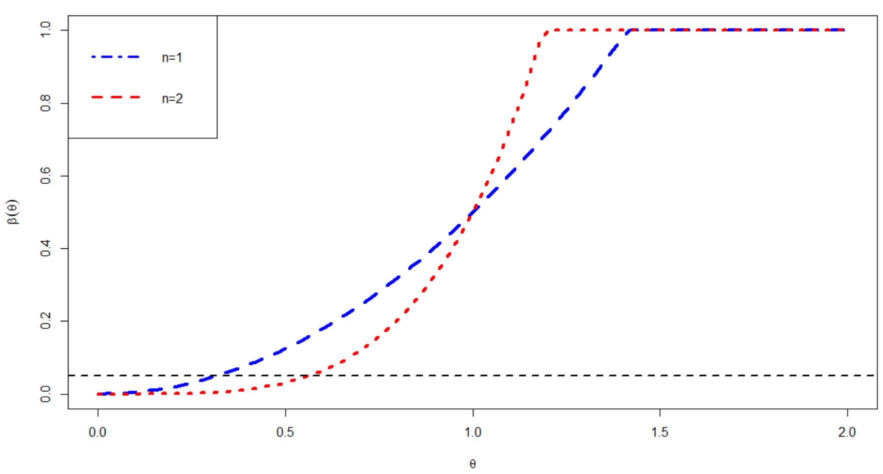
$$\beta_1(\theta) = \begin{cases} 1 \ , if \ \ \theta > \sqrt{2} \\ \frac{1}{2}\theta^2 \quad , if \ \ 0 < \theta \leq \sqrt{2} \end{cases},$$

$$\beta_1'(\theta) = \theta \ \ and \ \ \beta_1''(\theta) = 1$$

When n=2,

$$\beta_2(\theta) = \begin{cases} 1 \ , if \ \ \theta > 2^{\frac{1}{4}} \\ \frac{1}{2}\theta^4 \quad , if \ \ 0 < \theta \leq 2^{\frac{1}{4}} \end{cases},$$

$$\beta_2'(\theta) = 2\theta^3 \ \ and \ \ \beta_2''(\theta) = 6\theta^2$$



Rcodes:

```
library(ggplot2)
a=alpha<-0.5
b=theta0<-1
n<-c(1,2)
powerfunction<-function(theta,i=1){
  ifelse(theta>b*a^(-1/(2*n[i])),1,a*(theta/b)^(2*n[i]))
}

powerfunction1<-function(theta,i=2){
  ifelse(theta>b*a^(-1/(2*n[i])),1,a*(theta/b)^(2*n[i]))
}

curve(powerfunction(x),0,2,lty=2,lwd=4,col=4,xlab=expression(theta),ylab=expressio
n(beta(theta)))
curve(powerfunction1(x),0,2,add = T,lty=3,lwd=4,col=2)
abline(v=10,lty=2,lwd=2)
abline(h=0.05,lty=2,lwd=2)
legend("topleft",c("n=1","n=2"),col=c(4,2),lty=c(4,2),lwd=3)
```

(4)

$$\mathrm{R}(\theta_0) = \{\mathrm{X}|X_{(1)} \geq \theta_0\alpha^{-\frac{1}{2n}}\}$$

$$\Rightarrow \mathrm{A}(\theta_0) = \{\mathrm{X}|X_{(1)} < \theta_0\alpha^{-\frac{1}{2n}}\}$$

$$\Rightarrow \mathrm{C}(\mathrm{X}) = \left\{\theta\middle|X_{(1)} > \theta\alpha^{-\frac{1}{2n}}\right\} = \{\theta|X_{(1)}\alpha^{\frac{1}{2n}} < \theta\}$$

$$\Rightarrow \left(X_{(1)}\alpha^{\frac{1}{2n}}, \infty\right) is\ \ a\ \ (1-\alpha)\ \ CI\ \ for\ \ \theta$$

Exercise 9.54[p.461]

Let $X \sim n(\mu, \sigma^2)$, but now consider σ^2 unknown. For each $c \geq 0$, define an interval estimator for μ by $C(x) = [x - cs, x + cs]$, where s^2 is an estimator of σ^2 independent of X, $\frac{\nu S^2}{\sigma^2} \sim \chi_\nu^2$ (for example, the usual sample variance). Consider a modification of the loss in (9.3.4),

$$L((\mu, \sigma), C) = \frac{b}{\sigma} Length(C) - I_C(\mu)$$

(a) Show that the risk function, $R((\mu, \sigma), C)$, is given by

$$R((\mu, \sigma), C) = b(2cM) - [2P(T \leq c) - 1]$$

Where $T \sim t_\nu$ *and* $M = ES/\sigma$.

(b) If $b \leq \frac{1}{\sqrt{2\pi}}$, show that the c that minimizes the risk sastisfies

$$b = \frac{1}{\sqrt{2\pi}} (\frac{v}{v + c^2})^{\frac{(v+1)}{2}}$$

(c) Reconcile this problem with the known σ^2 case. Show that as $\nu \to \infty$, the solution here converges to the solution in the known σ^2 problem. (Be careful of the rescaling done to the loss function.)

Solution:

(a)

$$c(x) = [x - cS, x + cS], c \geq 0$$

$$\text{length}(c(x)) = 2cS$$

$$P_u(u \in c(x)) = P_u(x - cS \leq u \leq x + cS) = P_u(-c \leq \frac{x-u}{S} \leq c)$$

$$= P(T \leq c) - P(T \leq -c) = [2P(T \leq c) - 1]$$

$$R((u, \sigma), c) = E(L(u, \sigma), c) = E\left[\frac{b}{\sigma} 2cS - I_c(u)\right] = b2cM - E(I_c(u))$$

$$= b(2cM) - P_u(u \in c(x))$$

$$= b(2cM) - [2P(T \leq c) - 1], T \sim t_v \ \ and \ \ M = \frac{E(S)}{\sigma}$$

(b)

$$\text{Let } \mathrm{Y} = \frac{vS^2}{\sigma^2} \sim \chi_v^2$$

$$\mathrm{E}(\sqrt{Y}) = \int_0^\infty y^{\frac{1}{2}} \frac{1}{\Gamma(\frac{v+1}{2})2^{\frac{v+1}{2}}} y^{\frac{v+1}{2}} e^{\frac{-y}{2}} dy = \frac{\Gamma(\frac{v+1}{2})2^{\frac{1}{2}}}{\Gamma(\frac{v}{2})}$$

$$\mathrm{E}(\sqrt{Y}) = \mathrm{E}\left(\sqrt{\frac{vS^2}{\sigma^2}}\right) = \sqrt{v}E\left(\frac{S}{\sigma}\right) = \frac{\Gamma(\frac{v+1}{2})2^{\frac{v+1}{2}}}{\Gamma(\frac{v}{2})2^{\frac{v}{2}}}$$

$$\Rightarrow \mathrm{M} = \frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{\sqrt{2}}{\sqrt{v}}$$

Let t be p.d.f of t distribution

$$\frac{d}{dc}R\big((\mu,\sigma),C\big) = 2bM - 2t(c)$$

$$= 2\mathrm{b}\frac{\sigma\Gamma(\frac{v+1}{2})}{\sigma\Gamma(\frac{v}{2})}\frac{\sqrt{2}}{\sqrt{v}} - 2\frac{\Gamma(\frac{v+1}{2})}{\Gamma(\frac{v}{2})}\frac{1}{\sqrt{v\pi}}\frac{1}{(1+\frac{c^2}{v})^{\frac{v+1}{2}}}$$

$$= 2\sqrt{2}\frac{\Gamma(\frac{v+1}{2})}{\sqrt{v}\Gamma(\frac{v}{2})}[b - \frac{1}{\sqrt{2\pi}}(\frac{v}{v+c^2})^{\frac{v+1}{2}}]$$

If $\mathrm{b} > \frac{1}{\sqrt{2\pi}}, \frac{d}{dc}R\big((\mu,\sigma),C\big) > 0, \forall c, then\ \ R\big((\mu,\sigma),C\big)\ \ is\ \ minimize\ \ at\ \ c = 0$

If $\mathrm{b} \le \frac{1}{\sqrt{2\pi}}, \frac{d}{dc}R\big((\mu,\sigma),C\big) = 0 \Rightarrow \mathrm{b} = \frac{1}{\sqrt{2\pi}}(\frac{v}{v+c^2})^{\frac{(v+1)}{2}}$

$$\frac{d^2R}{dc^2} = 2\sqrt{2}\frac{\Gamma\left(\frac{v+1}{2}\right)}{\sqrt{v}\Gamma\left(\frac{v}{2}\right)}(-\frac{v+1}{2})\frac{1}{\sqrt{2\pi}}\left(\frac{v}{v+c^2}\right)^{\frac{(v+1)}{2}}(-v(v+c^2)^{-2}\cdot 2c)$$

$$= v^{-1}(v+1)\frac{c}{\sqrt{2\pi}}\left(\frac{v}{v+c^2}\right)^{\frac{(v+1)}{2}} > 0\ \ \forall \mathrm{c}$$

Therefore, if $\mathrm{b} \le \frac{1}{\sqrt{2\pi}}$, the c that minimizes the risk sastisfies $\mathrm{b} = \frac{1}{\sqrt{2\pi}}(\frac{v}{v+c^2})^{\frac{(v+1)}{2}}$.

(c)

First, we consider when σ^2 *is known*

$$\mathrm{c(x) = [x - c\sigma, x + c\sigma], c \geq 0}$$
$$\mathrm{length\big(c(x)\big) = 2c\sigma}$$

$$\mathrm{R\big((\mu), C\big) = E(L(\mu), C)) = E(}\frac{b}{\sigma}Length(C) - I_c(\mu))$$

$$= \frac{b}{\sigma}E(2c\sigma) - 1 \cdot P_\mu\big(u \in c(x)\big) = 2bc - P_\mu(\mu - c\sigma \leq x \leq \mu + c\sigma)$$

$$= 2\mathrm{bc} - P_\mu\left(-\mathrm{c} \leq \frac{x-\mu}{\sigma} \leq c\right) = 2bc - [2\Phi(c) - 1]$$

$$\frac{d}{dc}R\big((\mu,\sigma),C\big) = 2b - 2\phi(c) = 2[b - \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}]$$

If $\mathrm{b} > \frac{1}{\sqrt{2\pi}}, \frac{d}{dc}R\big((\mu),C\big) > 0, \forall c, then \;\; R\big((\mu,\sigma),C\big) \;\; is \;\; minimize \;\; at \;\; c = 0$

If $\mathrm{b} \leq \frac{1}{\sqrt{2\pi}}, \frac{d}{dc}R\big((\mu),C\big) = 0 \Rightarrow \mathrm{b} = \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}}$

$$\frac{d^2R}{dc^2} = 2c\phi(c) > 0 \;\; \forall \mathrm{c}$$

Then we consider the solution of (b) when $v \to \infty$,

$$\lim_{v\to\infty} b = \lim_{v\to\infty}\frac{1}{\sqrt{2\pi}}(\frac{v}{v+c^2})^{\frac{(v+1)}{2}} = \frac{1}{\sqrt{2\pi}}\lim_{v\to\infty}\exp\left(\frac{\ln\left(\frac{v}{v+c^2}\right)}{\left(\frac{2}{v+1}\right)}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp\left(\lim_{v\to\infty}\frac{\frac{v+c^2}{v}\left(\frac{c^2}{(v+c^2)^2}\right)}{-\frac{2}{(v+1)^2}}\right)$$

$$= \frac{1}{\sqrt{2\pi}}\exp(\lim_{v\to\infty} - \frac{c^2(v+1)^2}{2v^2 + 2vc^2})$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{c^2}{2}} \;\; \text{(By L'Hospital's Rule)}$$

Therefore, when $v \to \infty$, the solution of (b) converges to the solution in the known σ^2 problem.

Exercise 9.55[p.462]

The decision theoretic approach to set estimation can be quite useful (see Exercise 9.56) but it can also give some unsettling results, showing the need for thoughtful implementation. Consider again the case of $X \sim n(\mu, \sigma^2)$, σ^2 unknown, and suppose that we have an interval estimator for μ by $C(x) = [x - cs, x + cs]$, where s^2 is an estimator of σ^2 independent of $X, \frac{vS^2}{\sigma^2} \sim \chi^2_v$. This is, of course, the usual t interval, one of the great statistical procedures that has stood the test of time. Consider the

one of the great statistical procedures that has stood the test of time. Consider the

loss

$$L((\mu, \sigma), C) = b \ \ Length(C) - I_C(\mu)$$

Similar to that used in Exercise9.54, but without scaling the length. Construct another procedure C' as

$$C' = \begin{cases} [x - cs, x + cs] & if \ \ s \leq K \\ \emptyset & if \ \ s \geq K \end{cases}$$

Where K is a positive constant. Notice that C' does exactly the wrong thing. When s^2 is big and there is a lot of uncertainty, we would want the interval to be wide. But C' is empty! Show that we can find a value of K so that

$$R((\mu, \sigma), C') \leq R((\mu, \sigma), C) \ \ for \ \ every \ \ (\mu, \sigma)$$

With strict inequality for some (μ, σ).

Solution:

$$R((\mu, \sigma), C) = E[L(\mu, \sigma), C]$$
$$= E[L(\mu, \sigma), C|S < K]P(S < K) + E[L(\mu, \sigma), C|S > K]P(S > K)$$
$$= E[L(\mu, \sigma), C'|S < K]P(S < K) + E[L(\mu, \sigma), C|S > K]P(S > K)$$
$$= R((\mu, \sigma), C') + E[L(\mu, \sigma), C|S \geq K]P(S > K)$$

Where $C' = \emptyset, if \ \ s \geq K$

$$E[L(\mu, \sigma), C|S > K]P(S > K) = E(bLength(C) - I_C(\mu)|S > K)$$
$$= E(2bcS - I_C(\mu)|S > K)$$
$$> E(2bcK - 1|S > K) \ \ (since \ \ S > K \ \ and \ \ I_C(\mu) \leq 1)$$
$$= 2bcK - 1$$

Therefore , we can find a value of $K \geq \frac{1}{2bc}$ s.t

$$R((\mu, \sigma), C') \leq R((\mu, \sigma), C) \ \ for \ \ every \ \ (\mu, \sigma)$$