28/30

Final exam, Mathematical Statistics I, 2013 Fall (9:00-11:50)

Name: 何威霆

+4/4

1. Let $X_1,\ldots,X_n \overset{iid}{\sim} \mathrm{Exp}(\theta)$ with the pdf

$$f(x_i \mid \theta)=\begin{cases} 1/\theta \exp(-x_i/\theta) & \text{if } x_i \ge 0 \\ 0 & \text{if } x_i < 0 \end{cases}.$$

$E[\frac{\frac{1}{n}\sum X_i}{X_n}]$.

Derive $E\left[\frac{\bar{X}}{X_n}\right]$, where $\bar{X}=\frac{1}{n}\sum_{i=1}^{n} X_i$ (not only answer).

$\sum_{i=1}^{n} X_i \sim \mathrm{Gamma}(n, \beta = \frac{1}{\theta})$.

$E[\frac{\bar{X}}{X_n}] = E[\frac{\sum_{i=1}^{n} X_i}{nX_n}]$, we can indicate that nX_n is a sufficient statistic, $E[\frac{\sum_{i=1}^{n} X_i}{nX_n}]$

and $\sum_{i=1}^{n} X_i / nX_n$ is an ancillary. Then by Basu's theorem, $E[nX_n \cdot \frac{\sum_{i=1}^{n} X_i}{nX_n}] = E[nX_n]$.

$\therefore E[\sum_{i=1}^{n} X_i] = \theta n = E[nX_n] \cdot E[\frac{\bar{X}}{X_n}] = \frac{n}{\theta} E[\frac{\bar{X}}{X_n}] \Rightarrow E[\frac{\bar{X}}{X_n}] = 1$ #

+5/6

2. Let $X_1,\ldots,X_n \overset{iid}{\sim} N(\mu_X, 1)$ and $Y_1,\ldots,Y_n \overset{iid}{\sim} N(\mu_Y, 1)$, where $X_i \perp Y_i$ for $\forall i = 1,\ldots,n$. Suppose that (μ_X, μ_Y) are unknown. We wish to estimate an unknown parameter $\theta = P(X_1 \le Y_1)$.

$\frac{1}{n^2} n$

$X_1 - Y_1 \sim N(\mu_X - \mu_Y, 2)$

$\bar{X} \sim N(\mu_X, \frac{1}{n})$

1) Find a consistent estimator $\hat{\theta}$ of θ.
2) Prove the consistency of $\hat{\theta}$.
3) Derive the convergence in distribution of $\sqrt{n}(\hat{\theta}-\theta)$.

(1) $P(X_1 \le Y_1) = P(X_1 - Y_1 \le 0) = P\left(\frac{(X_1-Y_1)-(\mu_X-\mu_Y)}{\sqrt{2}} \le -\frac{(\mu_X-\mu_Y)}{\sqrt{2}}\right)$

$= \Phi\left[-\frac{(\mu_X-\mu_Y)}{\sqrt{2}}\right]$, where Φ is cdf of normal

Therefore, we can say that $\Phi\left[-\frac{(\bar{X}-\bar{Y})}{\sqrt{2}}\right]$ is the consistent estimator of θ, where $\bar{X} = \frac{1}{n}\sum X_i$, $\bar{Y} = \frac{1}{n}\sum Y_i$.

(2) $P(|\bar{X}-\mu_X| \ge \varepsilon) = P((\bar{X}-\mu_X)^2 \ge \varepsilon^2) \le \frac{E[(\bar{X}-\mu_X)^2]}{\varepsilon^2} = \frac{Var(\bar{X})}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \xrightarrow{n\to\infty}$

$P(|\bar{Y}-\mu_X| \ge \varepsilon) = P((\bar{Y}-\mu_Y) \ge \varepsilon^2) \le \frac{E[(\bar{Y}-\mu_Y)^2]}{\varepsilon^2} = \frac{Var(\bar{Y})}{\varepsilon^2} = \frac{1}{n\varepsilon^2} \xrightarrow{n\to\infty} 0$

Therefore $\bar{X} \xrightarrow{P} \mu_X$, $\bar{Y} \xrightarrow{P} \mu_Y$, then by Slutsky's theorem,

$\bar{X}-\bar{Y} \xrightarrow{P} \mu_X - \mu_Y$.

According to continuous mapping theorem, let $h(x) = -\frac{x}{\sqrt{2}}$ which continuous at x, so $-\frac{(\bar{X}-\bar{Y})}{\sqrt{2}} \xrightarrow{P} -\frac{(\mu_X-\mu_Y)}{\sqrt{2}}$. And since Φ, the cdf of normal is continuous for $(-\infty, \infty) \Rightarrow \Phi[-\frac{(\bar{X}-\bar{Y})}{\sqrt{2}}] \xrightarrow{P} \Phi[-\frac{(\mu_X-\mu_Y)}{\sqrt{2}}]$. #

(3)

By delta method,

if $\sqrt{n}(Y_n - \theta) \xrightarrow{d} N(0, \sigma^2)$, then $\sqrt{n}(g(Y_n) - g(\theta)) \xrightarrow{d} N(0, \{g'(\theta)\}^2 \sigma^2)$.

Now, $\hat{\theta} = \Phi\left[\frac{-(\bar{X} - \bar{Y})}{\sqrt{2}}\right] \xrightarrow{P} \Phi\left[-\frac{(\mu_X - \mu_Y)}{\sqrt{2}}\right] = \theta$.

Know that $\lim_{n\to\infty} P\left(-\frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{2}/\sqrt{n}} \le z\right) = P(Z \le z)$, that is

$\sqrt{n}\left(-\frac{(\bar{X}-\bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{2}}\right) \xrightarrow{d} N(0, 1)$ $\left[\text{i.e. } \sqrt{n}\left((\bar{Y}-\bar{X}) - (\mu_Y - \mu_X)\right) \xrightarrow{d} N(0, 2)\right]$

Therefore,

$$\sqrt{n}(\hat{\theta} - \theta) = \sqrt{n}\left[\Phi\left(-\frac{(\bar{X}-\bar{Y})}{\sqrt{2}}\right) - \Phi\left(-\frac{(\mu_X - \mu_Y)}{\sqrt{2}}\right)\right]$$

$$\xrightarrow{d} N\left(0, \left\{\Phi'\left(-\frac{\mu_X - \mu_Y}{\sqrt{2}}\right)\right\}^2\right). \checkmark$$

|| write the formula.

$$\Rightarrow \sqrt{n}(\hat{\theta} - \theta) \xrightarrow{d} N\left(0, \frac{(\mu_X - \mu_Y)^2}{2}\right) \#$$

$$\left\{\phi\left(\frac{-(\mu_X - \mu_Y)}{\sqrt{2}}\right)\right\}^2$$

$$= \frac{1}{2\pi} \exp\left\{-\frac{(\mu_X - \mu_Y)^2}{2}\right\}$$

Finish the calculation.

3. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\theta, \theta^2)$ for $\theta > 0$.

1) Rewrite the pdf of X_i in the form

$$f(x_i \mid \theta) = h(x_i)c(\theta)\exp\{ w_1(\theta)t_1(x_i) + w_2(\theta)t_2(x_i) \}$$

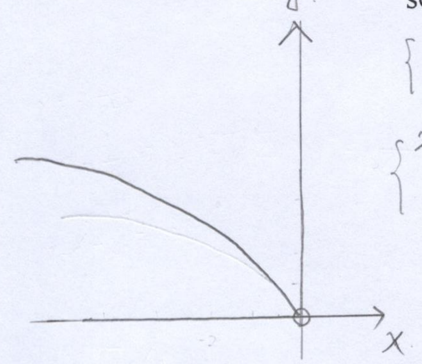
$$f(x_i \mid \theta) = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{1}{2\theta^2}(x_i-\theta)^2\right\} = \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{1}{2\theta^2}x_i^2 + \frac{1}{\theta}x_i - \frac{1}{2}\right\}$$

+5/5

$$= \frac{1}{\sqrt{2\pi\theta^2}} \exp\left\{-\frac{1}{2}\right\} \exp\left\{-\frac{1}{2\theta^2}x_i^2 + \frac{1}{\theta}x_i\right\} = h(x)c(\theta)\exp\left\{ \begin{matrix} W_1(\theta)t_1(x_i) \\ + W_2(\theta)t_2(x_i) \end{matrix} \right\}$$

where $h(x) = e^{-\frac{1}{2}}$, $c(\theta) = \frac{1}{\sqrt{2\pi\theta^2}}$, $W_1(\theta) = -\frac{1}{2\theta^2}$, $t_1(x_i) = x_i^2$, $W_2(\theta) = \frac{1}{\theta}$, $t_2(x_i) = x_i$ #

2) Show that the space $\{(w_1(\theta), w_2(\theta)) \mid \theta > 0\}$ does not contain any open set in R^2. [You may draw a figure to explain this]



$$\{(W_1(\theta), W_2(\theta)) \mid \theta > 0\} = \left\{\left(-\frac{1}{2\theta^2}, \frac{1}{\theta}\right) \middle| \theta > 0\right\}$$

$$\begin{cases} x = -\frac{1}{2\theta^2} \\ y = \frac{1}{\theta} \end{cases} \Rightarrow y^2 = -2x, \text{ for } y > 0,\ x < 0$$

By the graph, $\left\{\left(-\frac{1}{2\theta^2}, \frac{1}{\theta}\right) \middle| \theta > 0\right\}$ doesn't contain any open set in $\mathbb{R}^2$. #

4. Let $X_1, \ldots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$ for $\mu \in R$ and $\sigma^2 > 0$. Show that $(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2)$ is a complete and sufficient statistic for (μ, σ^2).

$x_i^2 - 2\mu x_i + \mu^2$

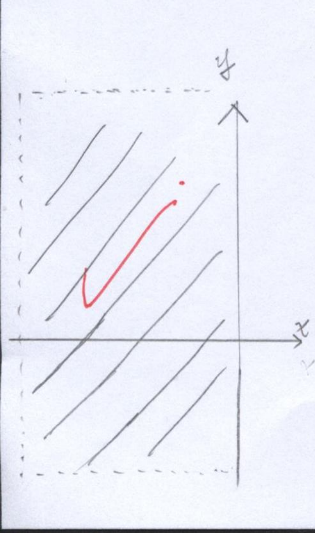
+5/5

$$\prod_{i=1}^n f(x_i \mid \mu, \sigma^2) = \prod_{i=1}^n (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2}(x_i-\mu)^2\right\}$$

$$= (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}\sum x_i^2 + \frac{1}{\sigma^2}\mu\sum x_i - \frac{n\mu^2}{2\sigma^2}\right\}$$

By Factorization theorem, letting $g(\underset{\sim}{\theta} \mid x_i) = \exp\left\{-\frac{1}{2\sigma^2}x_i^2 + \frac{1}{\sigma^2}\mu x_i\right\}$, $h(x) = (2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{n\mu^2}{2\sigma^2}\right\}$. Thus, $(\sum X_i, \sum X_i^2)$ is a sufficient statistic for (μ, σ^2).

And for completeness,



$$f(x_i \mid \mu, \sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left\{-\frac{1}{2\sigma^2}x_i^2 + \frac{1}{\sigma^2}\mu x_i - \frac{\mu^2}{2\sigma^2}\right\}$$

Let $h(x) = 1$, $c(\underset{\sim}{\theta}) = (2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{\mu^2}{2\sigma^2}\right\}$, $t_1(x_i) = x_i^2$, $W_1(\theta) = -\frac{1}{2\sigma^2}$

$t_2(x_i) = x_i$, $W_2(\underset{\sim}{\theta}) = \frac{\mu}{\sigma^2}$

$\left\{\left(-\frac{1}{2\sigma^2}, \frac{\mu}{\sigma^2}\right) \middle| \mu \in \mathbb{R}, \sigma^2 > 0\right\}$ contains all points $\begin{cases} x < 0 \\ y \in \mathbb{R} \end{cases}$, it's an open set on $\mathbb{R}^2$. Therefore, $(\sum X_i, \sum X_i^2)$ is a complete sufficient statistic for (μ, σ^2) #

+4/6

5. For a parameter θ, let $T(\mathbf{X})$ be a complete and sufficient statistic and $S(\mathbf{X})$ be an ancillary statistic. The corresponding pdf (or pmf) is denoted by $f_T(t\mid\theta)$ and $f_S(s)$, respectively.

1) State and prove Basu's Theorem for discrete case [i.e., $(T(\mathbf{X}), S(\mathbf{X}))$ has a joint pmf]. If $T(X)$ is a sufficient statistic, $S(X)$ is an ancillary, then $T(X) \perp S(X)$.

proof. Want to show that $P(S(X)=s \mid T(X)=t) = P(S(X)=s)$.

① $P(S(X)=s) = \sum_{t\in\mathcal{T}} P(S(X)=s \mid T(X)=t)\, P(T(X)=t)$.

② $P(S(X)=s) = \sum_{t\in\mathcal{T}} P(S(X)=s)\, P(T(X)=t)$, $\mathcal{T}$ is range of t.

①-② $\Rightarrow$ $0 = \sum \underbrace{[P(S(X)=s \mid T(X)=t) - P(S(X)=s)]}_{g(t)} P(T(X)=t) = E[g(t)]$ (given)

$\because$ $T(X)$ is C.S.S

$\therefore$ $g(t)=0 \Rightarrow P(S(X)=s \mid T(X)=t) = P(S(X)=s) \Rightarrow T(X) \perp S(X)$

2) State and prove Basu's Theorem for continuous case [i.e., $(T(\mathbf{X}), S(\mathbf{X}))$ has a joint pdf]. If $T(X)$ is a sufficient statistic, $S(X)$ is an ancillary, then $T(X) \perp S(X)$.

proof.

$P(S(X)=s) = \int_{t\in\mathcal{T}} P(S(X)=s \mid T(X)=t)\, P(T(X)=t)\, dt$

For continuouse case.

$P(S(X)=s) = \int_{t\in\mathcal{T}} P(S(X)=s)\, P(T(X)=t)\, dt$

$0 = \int_{t\in\mathcal{T}} \underbrace{[P(S(X)=s \mid T(X)=t) - P(S(X)=s)]}_{g(t)} P(T(X)=t)\, dt = E[g(t)]$

Since $T(X)$ is C.S.S, thus $g(t) = P(S(X)=s \mid T(X)=t) - P(S(X)=s) = 0$

$\therefore$ $P(S(X)=s \mid T(X)=t) = P(S(X)=s) \Rightarrow T(X) \perp S(X)$. #

+5/4

6. The pmf of the negative binomial distribution is

$$f(x\mid p)=\binom{x+r-1}{x}p^r(1-p)^x,\quad x=0,1,2....$$

Let $r=2$. Draw the detailed likelihood function given $X=2$, and indicate the maximum and minimum of the likelihood function.

Let $L(p)=f(2\mid p)=\binom{3}{2}p^2(1-p)^2=3p^2(1-p)^2$

$L'(p)=6p(1-p)^2-6p^2(1-p)$

$=6p(p^2-2p+1)-6p^2+6p^3$

$=12p^3-18p^2+6p$

$L''(p)=36p^2-36p+6$

$L(0)=0$

$L(\frac{1}{2})=3\cdot\frac{1}{4}\cdot\frac{1}{4}=\frac{3}{16}$

$L(1)=0$

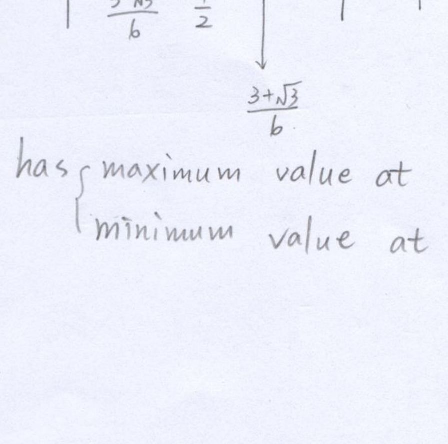
Let $L'(p)=12p^3-18p^2+6p=0$

$\Rightarrow 2p^3-3p^2+p=0 \Rightarrow p(2p-1)(p-1)=0 \Rightarrow p=0,\frac{1}{2},1$

Let $L''(p)=36p^2-36p+6=0 \Rightarrow 6p^2-6p+1=0 \Rightarrow p=\frac{3\pm\sqrt{3}}{6}$

p	0	$\frac{1}{2}$	1	$\frac{3+\sqrt{3}}{6}$	$\frac{3-\sqrt{3}}{6}$
$L'(p)$	0	0	0	<0	>0
$L''(p)$	>0	<0	>0	0	0
graph	$\cup$	$\cap$	$\cup$	╲	∫

$\Rightarrow$



(+1)

Therefore, the likelihood function has $\begin{cases}\text{maximum value at } p=\frac{1}{2}\\ \text{minimum value at } p=0,1\end{cases}$ #