Homework#3, Mathematical Statistics I, 2013 Fall

- A bivariate random vector (X,Y) is defined as $X = U + W$ and $Y = V + W$.

1. Calculate ρ_{XY} when $U,V \overset{iid}{\sim} Poisson(1)$, $W \sim Poisson(\tau)$ and $(U,V) \perp W$.
2. Calculate ρ_{XY} when $U,V \overset{iid}{\sim} N(0,1)$, $W \sim N(0,\tau^2)$ and $(U,V) \perp W$.
3. In the model as above, let $X = aU + bW$ and $Y = aV + bW$. Find (*a*, *b*) such that (X,Y) follows a bivariate normal with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$ and $\rho_{XY} = \rho$.

- Exercise 4.5

- In (a) of Exercise 4.5, derive

(i) $Cov(X,Y)$

(ii) ρ_{XY}.

10222501 黃筱涵 Huang Siao-Han

- A bivariate random vector (X, Y) is defined as $X = U + W$ and $Y = V + W$

1. $U, V \overset{iid}{\sim} \text{Poisson}(1)$, $W \sim \text{Poisson}(\tau)$ and $(U, V) \perp W$ calculate ρ_{XY}.

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

$$Cov(X,Y) = Cov(U+W, V+W)$$
$$= \cancelto{0}{Cov(U,V)} + \cancelto{0}{Cov(U,W)} + \cancelto{0}{Cov(V,W)} + Cov(W,W)$$

$\because U, V \overset{iid}{\sim} \text{Poisson}(1)$

$\therefore U \perp V \Rightarrow Cov(U,V) = 0$

$\because (U,V) \perp W$

$\therefore U \perp W$ and $V \perp W \Rightarrow Cov(U,W) = 0$, $Cov(V,W) = 0$

$$= Cov(W,W)$$
$$= Var(W) \quad (\because W \sim \text{Poisson}(\tau) \ \therefore Var(W) = \tau)$$
$$= \tau$$

$$Var(X) = Var(U+W)$$
$$= Var(U) + Var(W) + 2\cancelto{0}{Cov(U,W)}$$
$$= 1 + \tau \quad (\because U \sim \text{Poisson}(1) \ \therefore Var(U) = 1)$$

$$Var(Y) = Var(V+W)$$
$$= Var(V) + Var(W) + 2\cancelto{0}{Cov(V,W)}$$
$$= 1 + \tau \quad (\because V \sim \text{Poisson}(1) \ \therefore Var(V) = 1)$$

$$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$
$$= \frac{\tau}{\sqrt{1+\tau}\sqrt{1+\tau}}$$
$$= \frac{\tau}{1+\tau} \ \#$$

2. $U, V \overset{iid}{\sim} N(0,1)$, $W \sim N(0, \tau^2)$ and $(U,V) \perp W$ Calculate ρ_{XY}.

$$Cov(X,Y) = Cov(U+W, V+W)$$
$$= \cancelto{0}{Cov(U,V)} + \cancelto{0}{Cov(U,W)} + \cancelto{0}{Cov(V,W)} + Cov(W,W)$$

$\because U, V \overset{iid}{\sim} N(0,1)$

$\therefore U \perp V \Rightarrow Cov(U,V) = 0$

$\because (U,V) \perp W$

$\therefore U \perp W$ and $V \perp W \Rightarrow Cov(U,W) = 0$, $Cov(V,W) = 0$

$$= Cov(W,W)$$
$$= Var(W) \quad (\because W \sim N(0,\tau^2) \ \therefore Var(W) = \tau^2)$$
$$= \tau^2$$

$$Var(X) = Var(U+W)$$
$$= Var(U) + Var(W) + 2\cancelto{0}{Cov(U,W)}$$
$$= 1 + \tau^2 \quad (\because U \sim N(0,1) \ \therefore Var(U) = 1)$$

$$Var(Y) = Var(V+W)$$
$$= Var(V) + Var(W) + 2\cancelto{0}{Cov(V,W)}$$
$$= 1 + \tau^2 \quad (\because V \sim N(0,1) \ \therefore Var(V) = 1)$$

$$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$
$$= \frac{\tau^2}{\sqrt{1+\tau^2}\sqrt{1+\tau^2}}$$
$$= \frac{\tau^2}{1+\tau^2} \ \#$$

3. In the model as above, let $X = aU + bW$ and $Y = aV + bW$. Find (a, b) such that (X, Y) follows a bivariate normal with $\mu_X = \mu_Y = 0$, $\sigma_X = \sigma_Y = 1$ and $\rho_{XY} = \rho$.

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$$(X,Y) \sim N_2(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$$
$$\Rightarrow X \sim N(\mu_X, \sigma_X^2), \quad Y \sim N(\mu_X, \sigma_Y^2)$$
$$(X,Y) \sim N_2(0, 0, 1, 1, \rho)$$
$$\Rightarrow X \sim N(0,1), \quad Y \sim N(0,1)$$

$$Var(X) = Var(aU + bW)$$
$$= a^2 Var(U) + b^2 Var(W) + 2ab\,\cancelto{0}{Cov(U,W)} \quad (\because U \perp W)$$
$$= a^2 + b^2\tau^2 \quad (\because Var(U) = 1, \ Var(W) = \tau^2)$$
$$= 1 \quad (\because X \sim N(0,1) \ \therefore Var(X) = 1)$$

$$Var(Y) = Var(aV + bW)$$
$$= a^2 Var(V) + b^2 Var(W) + 2ab\,\cancelto{0}{Cov(V,W)} \quad (\because V \perp W)$$
$$= a^2 + b^2\tau^2 \quad (\because Var(V) = 1, \ Var(W) = \tau^2)$$
$$= 1 \quad (\because Y \sim N(0,1) \ \therefore Var(Y) = 1)$$

$$Cov(X,Y) = Cov(aU + bW, aV + bW)$$
$$= a^2\,\cancelto{0\,(\because U \perp V)}{Cov(U,V)} + ab\,\cancelto{0\,(\because U \perp W)}{Cov(U,W)} + ba\,\cancelto{0\,(\because W \perp V)}{Cov(W,V)} + b^2 Cov(W,W)$$
$$= b^2 Cov(W,W)$$
$$= b^2 Var(W)$$
$$= b^2\tau^2$$

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$
$$= \frac{b^2\tau^2}{a^2 + b^2\tau^2}$$
$$= \rho$$

$$\Rightarrow \begin{cases} a^2 + b^2\tau^2 = 1 \\ \dfrac{b^2\tau^2}{a^2 + b^2\tau^2} = \rho \end{cases}$$

$$\Rightarrow b^2\tau^2 = \rho$$
$$\Rightarrow b^2 = \frac{\rho}{\tau^2}$$
$$\Rightarrow b = \pm\sqrt{\frac{\rho}{\tau^2}}$$
$$\Rightarrow a^2 + \frac{\rho}{\tau^2} \times \tau^2 = 1$$
$$\Rightarrow a^2 = 1 - \rho$$
$$\Rightarrow a = \pm\sqrt{1-\rho}$$

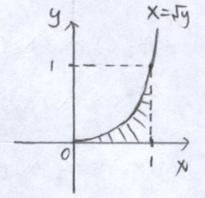
$$\therefore (a,b) = \left(\sqrt{1-\rho}, \sqrt{\frac{\rho}{\tau^2}}\right), \left(\sqrt{1-\rho}, -\sqrt{\frac{\rho}{\tau^2}}\right), \left(-\sqrt{1-\rho}, \sqrt{\frac{\rho}{\tau^2}}\right), \left(-\sqrt{1-\rho}, -\sqrt{\frac{\rho}{\tau^2}}\right) \ \#$$

- Exercise 4.5

(a) $f(x,y) = x+y$, $0 \le x \le 1$, $0 \le y \le 1$

Find $P(X > \sqrt{Y})$.

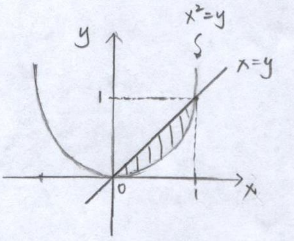
$$P(X>\sqrt{Y}) = \int_0^1 \int_{\sqrt{y}}^1 (x+y)\,dx\,dy$$
$$= \int_0^1 \left(\frac{x^2}{2}+yx\right)\Big|_{\sqrt{y}}^1 dy$$
$$= \int_0^1 \left(\frac{1}{2}+y\right)-\left(\frac{y}{2}+y\sqrt{y}\right) dy$$
$$= \int_0^1 \left(\frac{1}{2}+\frac{y}{2}-y^{\frac{3}{2}}\right) dy$$
$$= \left(\frac{y}{2}+\frac{y^2}{4}-\frac{y^{\frac{5}{2}}}{\frac{5}{2}}\right)\Big|_0^1$$
$$= \frac{1}{2}+\frac{1}{4}-\frac{2}{5}$$
$$= \frac{7}{20} \ \#$$



(b) $f(x,y) = 2x$, $0 \le x \le 1$, $0 \le y \le 1$

Find $P(X^2 < Y < X)$.

$$P(X^2<Y<X) = \int_0^1 \int_{x^2}^{x} 2x\,dy\,dx$$
$$= \int_0^1 (2xy)\Big|_{x^2}^{x} dx$$
$$= \int_0^1 2x(x-x^2)\,dx$$
$$= \int_0^1 2x^2-2x^3\,dx$$
$$= 2\frac{x^3}{3}-2\frac{x^4}{4}\Big|_0^1$$
$$= \frac{2}{3}-\frac{1}{2}$$
$$= \frac{1}{6} \ \#$$



- In (a) of Exercise 4.5, derive

(i) Cov(X, Y)

$$Cov(X,Y) = E(XY)-E(X)E(Y)$$
$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\,f(x,y)\,dx\,dy$$
$$= \int_0^1\int_0^1 xy(x+y)\,dx\,dy$$
$$= \int_0^1\int_0^1 x^2y+xy^2\,dx\,dy$$
$$= \int_0^1 \left[\left(\frac{x^3}{3}\right)y+\left(\frac{x^2}{2}\right)y^2\right]\Big|_0^1 dy$$
$$= \int_0^1 \left(\frac{y}{3}+\frac{y^2}{2}\right)dy$$
$$= \left(\frac{y^2}{6}+\frac{y^3}{6}\right)\Big|_0^1$$
$$= \frac{1}{6}+\frac{1}{6}$$
$$= \frac{1}{3}$$

$$f(x) = \int_{-\infty}^{\infty} f(x,y)\,dy$$
$$= \int_0^1 (x+y)\,dy$$
$$= x+\left(\frac{y^2}{2}\right)\Big|_0^1$$
$$= x+\frac{1}{2} \ , \ 0 \le x \le 1.$$

Similarly, $f(y) = \int_0^1 (x+y)\,dx$
$$= y+\frac{1}{2} \ , \ 0 \le y \le 1$$

$$E(X) = \int_{-\infty}^{\infty} x f(x)\,dx$$
$$= \int_0^1 x\left(x+\frac{1}{2}\right) dx$$
$$= \int_0^1 \left(x^2+\frac{1}{2}x\right) dx$$
$$= \left(\frac{x^3}{3}+\frac{x^2}{4}\right)\Big|_0^1$$
$$= \frac{1}{3}+\frac{1}{4}$$
$$= \frac{7}{12}$$

Similarly, $E(Y) = \int_0^1 y\left(y+\frac{1}{2}\right)dy = \frac{7}{12}$

$$\therefore Cov(X,Y) = E(XY)-E(X)E(Y)$$
$$= \frac{1}{3}-\frac{7}{12}\times\frac{7}{12}$$
$$= \frac{1}{3}-\frac{49}{144}$$
$$= \frac{-1}{144} \ \#$$

(ii) ρ_{XY}

$$\rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$
$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x)\,dx$$
$$= \int_0^1 x^2\left(x+\frac{1}{2}\right)dx$$
$$= \int_0^1 \left(x^3+\frac{x^2}{2}\right)dx$$
$$= \left(\frac{x^4}{4}+\frac{x^3}{6}\right)\Big|_0^1$$
$$= \frac{1}{4}+\frac{1}{6}$$
$$= \frac{5}{12}$$

Similarly, $E(Y^2) = \int_0^1 y^2\left(y+\frac{1}{2}\right)dy = \frac{5}{12}$

$$\therefore Var(X) = E(X^2)-(E(X))^2$$
$$= \frac{5}{12}-\left(\frac{7}{12}\right)^2$$
$$= \frac{5}{12}-\frac{49}{144}$$
$$= \frac{11}{144}$$

Similarly, $Var(Y) = E(Y^2)-(E(Y))^2$
$$= \frac{5}{12}-\left(\frac{7}{12}\right)^2$$
$$= \frac{11}{144}$$

$$\therefore \rho_{XY} = \frac{Cov(X,Y)}{\sigma_X\sigma_Y}$$
$$= \frac{\frac{-1}{144}}{\sqrt{\frac{11}{144}}\cdot\sqrt{\frac{11}{144}}}$$
$$= \frac{-1}{144}\times\frac{144}{11}$$
$$= \frac{-1}{11} \ \#$$