Mathematical Statistics I, 2013 Fall, HW#1

1. Sample space, random variable

Consider 4 coin tosses. For example, HTTH indicates the sequence of 4 successive results of Head or Tail.

1) Define the sample space S.

2) Let X_j, $j = 1,2,3,4$ be the cumulative number of Head's up to j^{th} tosses. Find the range of X_j, denoted by χ_j.

3) Make tables that enumerate the values of X_j, $j = 1,2,3,4$ (see Example 1.4.3).

4) Consider a random variable $Y = X_4 - \min_{j=1,2,3} X_j$. Make a table that enumerates the values of Y.

5) Assuming a "fair" coin, derive the cdf of Y and draw its graph (like Figure 1.5.1).

2. In Problem 3.36:

1) Calculate the mean and standard deviation of the random variable with the pdf

$$(1/\sigma) f((x-\mu)/\sigma).$$

2) Draw the graphs of (a), (b), (c) including the locations of the mean.

3. For the binomial distribution, calculate

1) Expectation, Variance and mgf
2) Skewness (p.79)
3) Kurtosis (p.79)

Mathematical Statistics I, HW #1.

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13

1.

(1) $S = \{(HHHH),$
$(THHH), (HTHH), (HHTH), (HHHT),$
$(TTHH), (THTH), (THHT), (HTTH),$
$(HTHT), (HHTT),$
$(TTTH), (TTHT), (THTT), (HTTT),$
$(TTTT)\}$

(2) X_j = the cumulative number of Head's up to j^{th} tosses. $j = 1, 2, 3, 4$.

The range for X_1 is $\mathcal{X}_1 = \{0, 1\}$.

The range for X_2 is $\mathcal{X}_2 = \{0, 1, 2\}$.

The range for X_3 is $\mathcal{X}_3 = \{0, 1, 2, 3\}$.

The range for X_4 is $\mathcal{X}_4 = \{0, 1, 2, 3, 4\}$.

(3).

S	HHHH	HTHH	HHTH	HHHT	THHH	HHTT	HTHT
X_1	1	1	1	1	0	1	1
X_2	2	1	2	2	1	2	1
X_3	3	2	2	3	2	2	2
X_4	4	3	3	3	3	2	2

S	HTTH	THHT	THTH	TTHH	TTTH	TTHT	THTT
X_1	1	0	0	0	0	0	0
X_2	1	1	1	0	0	0	1
X_3	1	2	1	1	0	1	1
X_4	2	2	2	2	1	1	1

S	HTTT	TTTT
X_1	1	0
X_2	1	0
X_3	1	0
X_4	1	0

(4)

$Y = X_4 - \min_{j=1,2,3} X_j$

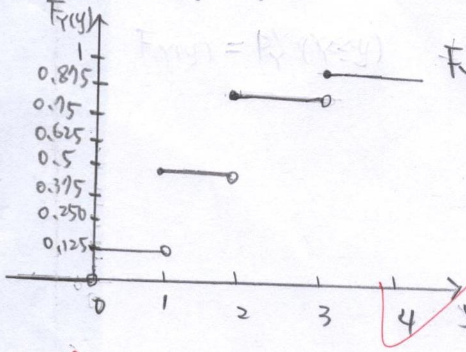
10/10.

S	HHHH	HTHH	HHTH	HHHT	THHH
Y	3	2	2	2	3

S	HHTT	HTHT	HTTH	THHT	THTH	TTHH	TTTH
Y	1	1	1	2	2	2	1

S	TTHT	THTT	HTTT	TTTT
Y	1	1	0	0

(5) cdf of Y $\quad F_{Y}(y) = P_Y(Y \le y)$



$$F_{Y}(y) = \begin{cases} 0, & \text{if } x < 0 \\ \frac{2}{16} = \frac{1}{8}, & \text{if } 0 \le x < 1 \\ \frac{8}{16} = \frac{1}{2}, & \text{if } 1 \le x < 2 \\ \frac{14}{16} = \frac{7}{8}, & \text{if } 2 \le x < 3 \\ 1, & \text{if } x \ge 3 \end{cases}$$

13

2. $f(x) = \frac{63}{4}(x^6 - x^8)$, $-1 < x < 1$

(1) For (a) $\mu = 0$, $\sigma = 1$, then $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = f(x)$

$E(X) = \int_{-1}^{1} x \frac{63}{4}(x^6 - x^8)\,dx = \frac{63}{4}\int_{-1}^{1}(x^7 - x^9)\,dx$

$= \frac{63}{4}\left[\int_{-1}^{1} x^7 dx - \int_{-1}^{1} x^9 dx\right]$

$= \frac{63}{4}\left[\frac{1}{8}x^8\Big|_{-1}^{1} - \frac{1}{10}x^{10}\Big|_{-1}^{1}\right] = 0$

$E(X^2) = \int_{-1}^{1} x^2 \frac{63}{4}(x^6 - x^8)\,dx = \frac{63}{4}\int_{-1}^{1}(x^8 - x^{10})\,dx$

$= \frac{63}{4}\left[\int_{-1}^{1} x^8 dx - \int_{-1}^{1} x^{10} dx\right]$

$= \frac{63}{4}\left[\frac{1}{9}x^9\Big|_{-1}^{1} - \frac{1}{11}x^{11}\Big|_{-1}^{1}\right]$

$= \frac{63}{4}\left[\frac{2}{9} - \frac{2}{11}\right] = \frac{63}{4}\left[\frac{4}{99}\right] = \frac{7}{11}$

$Var(X) = E(X^2) - E(X) = \frac{7}{11}$

$\sigma_X = \sqrt{\frac{7}{11}}$

For (b) $\mu = 3$, $\sigma = 1$, then $\frac{1}{\sigma} f\left(\frac{x-\mu}{\sigma}\right) = f(x-3)$.

Let X be a random variable with pdf $f(x)$; by (a) $E(X) = 0$, and $Var(X) = \frac{7}{11}$ exist. From Thm 3.5.1, if Z is a random variable with pdf $\left(\frac{1}{1} f\left(\frac{x-3}{1}\right)\right)$, then

$E(Z) = \sigma E(X) + \mu = 1 \cdot 0 + 3 = 3$

$Var(Z) = \sigma^2 Var(X) = 1 \cdot \frac{7}{11} = \frac{7}{11}$

$\sigma_Z = \sqrt{\frac{7}{11}}$

For (c), $\mu=3, \sigma=2$, then $\frac{1}{\sigma} f(\frac{x-\mu}{\sigma}) = \frac{1}{2} f(\frac{x-3}{2})$.

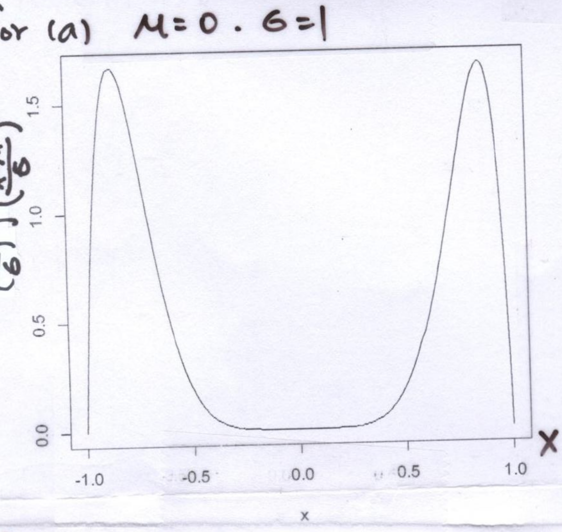
Let x be a random variable with pdf $f(x)$, by (a) $E(x)=0$ and $Var(x) = \frac{7}{11}$ exist. From Thm 3.5.7, if Y is a random variable with pdf $(\frac{1}{2} f(\frac{x-3}{2}))$, then

$E(Y) = \sigma E(x) + \mu = 2 \cdot 0 + 3 = 3$

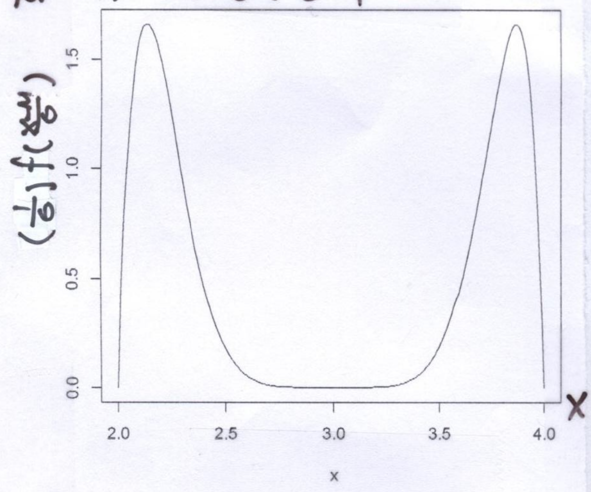
$Var(Y) = \sigma^2 Var(x) = 4 \cdot \frac{7}{11} = \frac{28}{11}$

$\sigma_Y = \sqrt{\frac{28}{11}}$

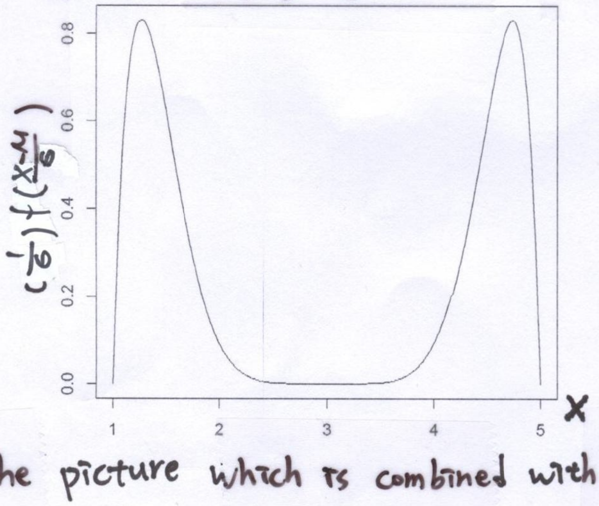
(2) for (a) $\mu = 0$. $\sigma = 1$



for (b) $\mu = 3$. $\sigma = 1$



for (c) $\mu = 3$ $\sigma = 2$



The picture which is combined with those graph and Rcode are on the last page.

3. Binomial distribution

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(1) Let $X \sim Bin(n, p)$ $\qquad (a+b)^r = \sum_{k=0}^{r} C_k^r a^k b^{r-k}$.

$P(x) = C_x^n p^x (1-p)^{n-x}$, $x = 0, 1, 2, 3, \dots, n$

$M_X(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} C_x^n p^x (1-p)^{n-x}$

$= \sum_{x=0}^{n} C_x^n (pe^t)(1-p)^{n-x} = (pe^t + 1 - p)^n$

$E(x) = \sum_{x=0}^{n} x \, C_x^n p^x (1-p)^{n-x}$

$= n \sum_{x=1}^{n} C_{x-1}^{n-1} p^x (1-p)^{n-x}$

Let $k = x-1$

$= np \sum_{k=0}^{n-1} C_k^{n-1} p^k (1-p)^{(n-1)-k}$

$= np \, [p + (1-p)]^{n-1} = np$

$E(x^2) = \sum_{x=0}^{n} x^2 C_x^n p^x (1-p)^{n-x}$

$= \sum_{x=1}^{n} n x C_{x-1}^{n-1} p^x (1-p)^{n-x}$

Let $k = x-1$

$= n \sum_{k=0}^{n-1} (k+1) C_k^{n-1} p^{k+1} (1-p)^{n-1-k}$

$= np \sum_{k=0}^{n-1} k C_k^{n-1} p^{k+1} (1-p)^{n-1-k} + np \sum_{k=0}^{n-1} C_k^{n-1} p^{k+1} (1-p)^{n-1-k}$

$= np(n-1)p + np = n(n-1)p^2 + np$

$Var(x) = E(x^2) - [E(x)]^2$

$= n(n-1)p^2 + np - n^2 p^2$

$= \cancel{n^2 p^2} - np^2 + np - \cancel{n^2 p^2} = np(1-p)$

(2).

skewness $= \frac{\mu_3}{(\mu_2)^{\frac{3}{2}}} = \frac{\mu^3}{\sigma^3}$

$K_X(t) = \ln M_X(t) = \ln (pe^t + 1 - p)^n$

$\mu_3 = \frac{d^3}{dt^3} K_X(t) \Big|_{t=0}$

$= \frac{d^2}{dt^2} n \frac{pe^t}{pe^t + 1 - p} \Big|_{t=0}$

$= \frac{d}{dt} \frac{[npe^t \cdot (pe^t + 1 - p)] - [npe^t \cdot pe^t]}{(pe^t + 1 - p)^2} \Bigg|_{t=0}$

$= \frac{[(npe^t - np^2e^t)(pe^t + 1 - p)^2] - [(npe^t - np^2e^t)(2(pe^t+1-p)pe^t)]}{(pe^t + 1 - p)^4}$

$= \frac{[(np - np^2)(p + 1 - p)^2] - [(np - np^2)(2(p+1-p)p)]}{(p + 1 - p)^4}$

$= np(1-p)(1-2p)$

$$\sigma^3 = np(1-p)\sqrt{np(1-p)}$$

$$\text{Skewness} = \frac{\mu_3}{\sigma^3} = \frac{\cancel{np(1-p)}(1-2p)}{\cancel{np(1-p)}\sqrt{np(1-p)}}$$

$$= \frac{1-2p}{\sqrt{np(1-p)}} \quad \#$$

(4) $\text{kurtosis} = \dfrac{\mu_4}{\mu_2^2} = \dfrac{\mu_4}{\sigma^4}$.

$$\mu_4 = k_4 + 3k_2^2$$

$$k_X(t) = \ln M_X(t) = \ln (pe^t + 1 - p)^n$$

$$k_2 = \frac{d^2}{dt^2} k_X(t) \Big|_{t=0}$$

$$= \frac{[npe^t \cdot (pe^t + 1 - p)] - [npe^t \cdot pe^t]}{(pe^t + 1 - p)^2} \Bigg|_{t=0}$$

$$= \frac{[np] - [np^2]}{1} = np(1-p)$$

$$k_4 = \frac{d^4}{dt^4} k_X(t) \Big|_{t=0}$$

$$= \frac{d}{dt} \frac{[(npe^t - np^2e^{t})(pe^t+1-p)^2] - [(npe^t - np^2e^t)(2(pe^t+1-p)pe^t)]}{(pe^t+1-p)^4} \Bigg|_{t=0}$$

$$= \frac{d}{dt} \frac{(npe^t - np^2e^t)\cancel{(pe^t+1-p)}(1-p-pe^t)}{\cancel{(pe^t+1-p)}^4} \Bigg|_{t=0}$$

$$= \frac{d}{dt} \frac{(npe^t - np^2e^t)(1-p-pe^t)}{(pe^t+1-p)^3} \Bigg|_{t=0}$$

$$= \frac{d}{dt} \frac{npe^t - np^2e^t - np^2e^t + np^3e^t - np^2e^{2t} + np^3e^{2t}}{(pe^t+1-p)^3} \Bigg|_{t=0}$$

$$= \frac{[(npe^t - 2np^2e^t + np^3e^t - 2np^2e^{2t} + 2np^3e^{2t})(pe^t+1-p)^3] - [(npe^t - 2np^2e^t + np^3e^t - np^2e^{2t} + np^3e^{2t})\,3(pe^t+1-p)^2pe^t]}{(pe^t+1-p)^6} \Bigg|_{t=0}$$

$$= (np - 2np^2 + np^3 - 2np^2 + 2np^3) - [3p(np - 2np^2 + np^3 - np^2 + np^3)]$$

$$= 3np^3 - 4np^2 + np - 3np^2 + 6np^3 - 3np^4 + 3np^3 - 3np^4$$

$$= 12np^3 - 7np^2 - 6np^4 + np$$

$$= np(12p^2 - 7p - 6p^3 + 1)$$

$$\mu_4 = k_4 + 3k_2^2$$

$$= np(12p^2 - 7p - 6p^3 + 1) + np(3np(1-p)^2)$$

$$\text{Kurtosis} = \frac{\mu_4}{\sigma^4}$$

$$= \frac{np(12p^2 - 7p - 6p^3 + 1) + np(3np(1-p)^2)}{np(np(1-p)^2)}$$

$$= 3 + \frac{(1-p)(1-6p+6p^2)}{np(1-p)^2}$$

$$= 3 + \frac{1 - 6p(1-p)}{np(1-p)} \quad \#$$

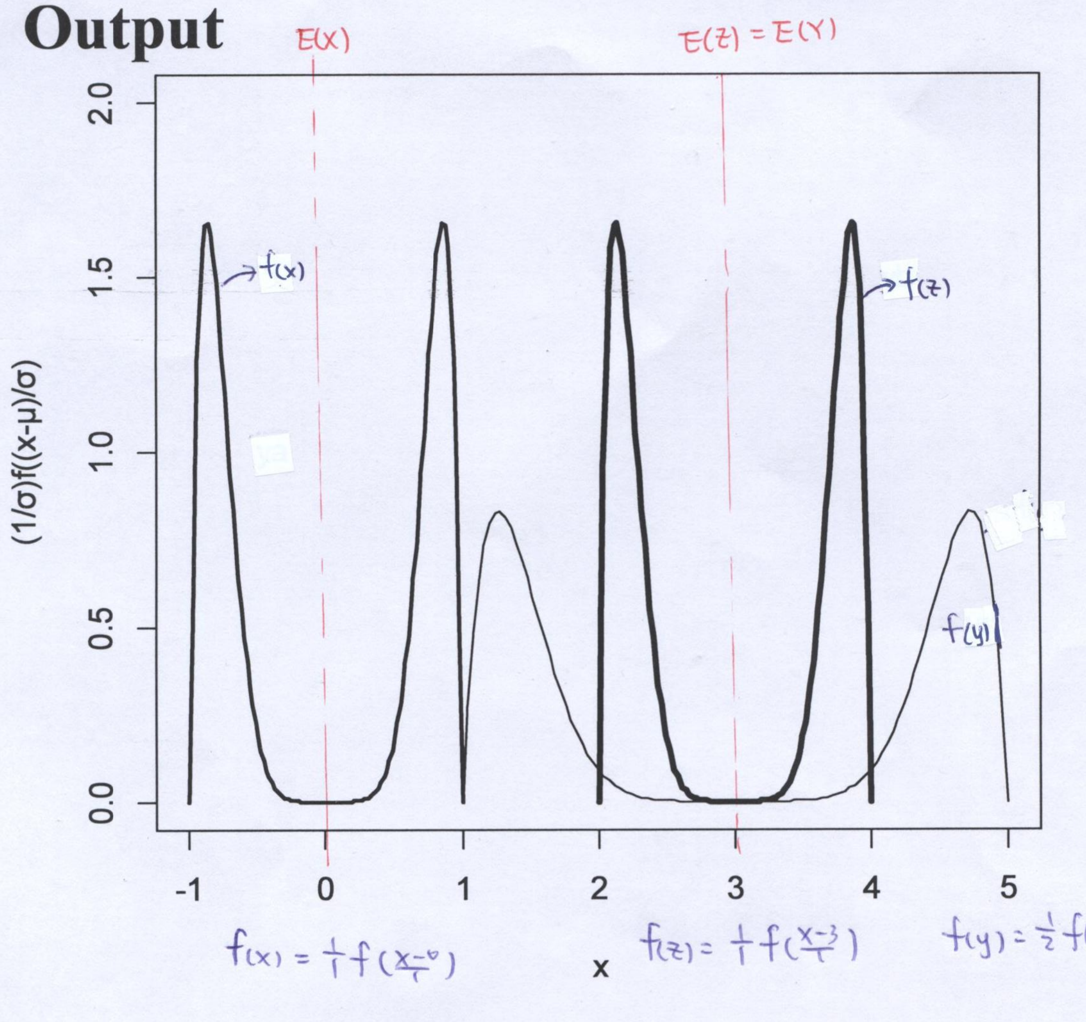
R code

```
fa = function(x){
1/1*(63/4*(((x-0)/1)^6-((x-0)/1)^8))     # μ = 0, σ = 1
}
curve( fa,-1,1, xlim = c(-1,5), ylim = c(0,2),ylab="(1/σ)f((x-μ)/σ)", lwd = 2)
text( -0.4, 1, "f(x)", lwd = 2 )

fb = function(x){
1/1*(63/4*(((x-3)/1)^6-((x-3)/1)^8))  # μ = 3 , σ = 1
}
curve( fb,2,4, add = TRUE, lwd = 3 )
text( 4.2, 1.5, "f(z)", lwd = 3 )

fc = function(x){
1/2*(63/4*(((x-3)/2)^6-((x-3)/2)^8))  # μ = 3, σ = 2 .
}
curve( fc, 1, 5, add = TRUE, lwd = 1 )
text( 4.8, 0.5, "f(y)" , lwd = 1 )
```

Output



$f(x) = \frac{1}{1} f\left(\frac{x-0}{1}\right)$ $f(z) = \frac{1}{1} f\left(\frac{x-3}{1}\right)$ $f(y) = \frac{1}{2} f\left(\frac{x-3}{2}\right)$