High-dimensional data analysis HW#4

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1. Draw cubic B-spline bases (order4) with $\xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta$

Order4(M=4)

Let $\begin{cases} K = 1 \quad M = 4 \\ \xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta \end{cases}$

K+2M=1+8=9

Augmented knots are

$\tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le \tau_5 \le \tau_6 \le \tau_7 \le \tau_8 \le \tau_9$ s.t.

1) $\tau_1 = \tau_2 = \tau_3 = \tau_4 = \xi_0 = 0$

2) $\tau_5 = \xi_1 = \Delta$

3) $\tau_6 = \tau_7 = \tau_8 = \tau_9 = \xi_2 = 2\Delta$

- Note: In order to be equally spaced $(\Delta = \xi_1 - \xi_0 = \xi_2 - \xi_1)$, I use $\Delta = 1$ in R program.

m=1 K+2M-1=1+8-1=8

$B_{i,1}(x) = I(\tau_i \le x < \tau_{i+1}), i = 1,2,3,4,5,6,7,8$

$B_{1,1}(x) = 0 \quad \because \tau_1 = \tau_2$

$B_{2,1}(x) = 0 \quad \because \tau_2 = \tau_3$

$B_{3,1}(x) = 0 \quad \because \tau_3 = \tau_4$

$B_{4,1}(x) = I(0 \le x < \Delta)$

$B_{5,1}(x) = I(\Delta \le x < 2\Delta)$

$B_{6,1}(x) = 0 \quad \because \tau_6 = \tau_7$

$B_{7,1}(x) = 0 \quad \because \tau_7 = \tau_8$

$B_{8,1}(x) = 0 \quad \because \tau_8 = \tau_9$

m=2 K+2M-m=1+8-2=7

$$B_{i,2}(x) = \frac{x-\tau_i}{\tau_{i+1}-\tau_i} B_{i,1}(x) + \frac{\tau_{i+2}-x}{\tau_{i+2}-\tau_{i+1}} B_{i+1,1}(x), i = 1,2,3,4,5,6,7$$

$$B_{1,2}(x) = \frac{x-\tau_1}{\tau_2-\tau_1} B_{1,1}(x) + \frac{\tau_3-x}{\tau_3-\tau_2} B_{2,1}(x)$$

$$= \frac{x-\tau_1}{\tau_2-\tau_1} * 0 + \frac{\tau_3-x}{\tau_3-\tau_2} * 0$$

$$= 0$$

$$B_{2,2}(x) = \frac{x-\tau_2}{\tau_3-\tau_2} B_{2,1}(x) + \frac{\tau_4-x}{\tau_4-\tau_3} B_{3,1}(x)$$

$$= \frac{x-\tau_2}{\tau_3-\tau_2} * 0 + \frac{\tau_4-x}{\tau_4-\tau_3} * 0$$

$$= 0$$

$$B_{3,2}(x) = \frac{x-\tau_3}{\tau_4-\tau_3} B_{3,1}(x) + \frac{\tau_5-x}{\tau_5-\tau_4} B_{4,1}(x)$$

$$= \frac{x-\tau_3}{\tau_4-\tau_3} * 0 + \frac{\tau_5-x}{\tau_5-\tau_4} I(0 \le x < \Delta)$$

$$= \frac{\Delta - x}{\Delta} I(0 \le x < \Delta)$$

$$B_{4,2}(x) = \frac{x-\tau_4}{\tau_5-\tau_4} B_{4,1}(x) + \frac{\tau_6-x}{\tau_6-\tau_5} B_{5,1}(x)$$

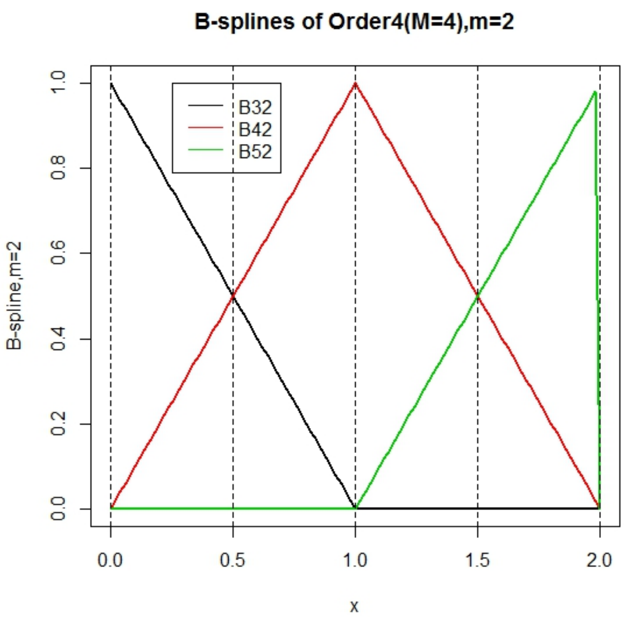
$$= \frac{x-\tau_4}{\tau_5-\tau_4} I(0 \le x < \Delta) + \frac{\tau_6-x}{\tau_6-\tau_5} I(\Delta \le x < 2\Delta)$$

$$= \frac{x}{\Delta} I(0 \le x < \Delta) + \frac{2\Delta - x}{\Delta} I(\Delta \le x < 2\Delta)$$

$$B_{5,2}(x) = \frac{x-\tau_5}{\tau_6-\tau_5} B_{5,1}(x) + \frac{\tau_7-x}{\tau_7-\tau_6} B_{6,1}(x)$$

$$= \frac{x-\tau_5}{\tau_6-\tau_5} I(\Delta \le x < 2\Delta) + \frac{\tau_7-x}{\tau_7-\tau_6} * 0$$

$$= \frac{x-\Delta}{\Delta} I(\Delta \le x < 2\Delta)$$



$$B_{6,2}(x) = \frac{x-\tau_6}{\tau_7-\tau_6} B_{6,1}(x) + \frac{\tau_8-x}{\tau_8-\tau_7} B_{7,1}(x)$$

$$= \frac{x-\tau_6}{\tau_7-\tau_6} * 0 + \frac{\tau_8-x}{\tau_8-\tau_7} * 08$$

$$= 0$$

$$B_{7,2}(x) = \frac{x-\tau_7}{\tau_8-\tau_7} B_{7,1}(x) + \frac{\tau_9-x}{\tau_9-\tau_8} B_{8,1}(x)$$

$$= \frac{x-\tau_7}{\tau_8-\tau_7} * 0 + \frac{\tau_9-x}{\tau_9-\tau_8} * 0$$

$$= 0$$

m=3 K+2M-m=1+8-3=6

$$\mathrm{B}_{i,3}(x)=\frac{x-\tau_i}{\tau_{i+2}-\tau_i}\mathrm{B}_{i,2}(x)+\frac{\tau_{i+3}-x}{\tau_{i+3}-\tau_{i+1}}\mathrm{B}_{i+1,2}(x), i=1,2,3,4,5,6$$

$$\begin{aligned}\mathrm{B}_{1,3}(x)&=\frac{x-\tau_1}{\tau_3-\tau_1}\mathrm{B}_{1,2}(x)+\frac{\tau_4-x}{\tau_4-\tau_2}\mathrm{B}_{2,2}(x)\\&=\frac{x-\tau_1}{\tau_3-\tau_1}*0+\frac{\tau_4-x}{\tau_4-\tau_2}*0\\&=0\end{aligned}$$

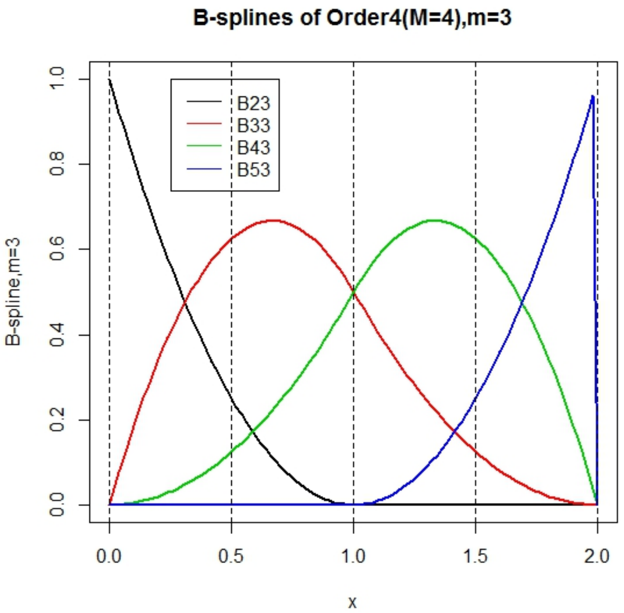
$$\begin{aligned}\mathrm{B}_{2,3}(x)&=\frac{x-\tau_2}{\tau_4-\tau_2}\mathrm{B}_{2,2}(x)+\frac{\tau_5-x}{\tau_5-\tau_3}\mathrm{B}_{3,2}(x)\\&=\frac{x-\tau_2}{\tau_4-\tau_2}*0+\frac{\tau_5-x}{\tau_5-\tau_3}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]\\&=\frac{\Delta-x}{\Delta}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]\\&=\left(\frac{\Delta-x}{\Delta}\right)^2 I(0\le x<\Delta)\end{aligned}$$

$$\begin{aligned}\mathrm{B}_{3,3}(x)&=\frac{x-\tau_3}{\tau_5-\tau_3}\mathrm{B}_{3,2}(x)+\frac{\tau_6-x}{\tau_6-\tau_4}\mathrm{B}_{4,2}(x)\\&=\frac{x-\tau_3}{\tau_5-\tau_3}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]+\frac{\tau_6-x}{\tau_6-\tau_4}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]\\&=\frac{x}{\Delta}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]+\frac{2\Delta-x}{2\Delta}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]\\&=\left[\frac{x(\Delta-x)}{\Delta^2}+\frac{x(2\Delta-x)}{2\Delta^2}\right]I(0\le x<\Delta)+\frac{(2\Delta-x)^2}{2\Delta^2}I(\Delta\le x<2\Delta)\\&=\left(\frac{-3x^2+4x\Delta}{2\Delta^2}\right)I(0\le x<\Delta)+\frac{(2\Delta-x)^2}{2\Delta^2}I(\Delta\le x<2\Delta)\end{aligned}$$

$$\begin{aligned}\mathrm{B}_{4,3}(x)&=\frac{x-\tau_4}{\tau_6-\tau_4}\mathrm{B}_{4,2}(x)+\frac{\tau_7-x}{\tau_7-\tau_5}\mathrm{B}_{5,2}(x)\\&=\frac{x-\tau_4}{\tau_6-\tau_4}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]+\frac{\tau_7-x}{\tau_7-\tau_5}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]\\&=\frac{x}{2\Delta}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]+\frac{2\Delta-x}{\Delta}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]\\&=\frac{x^2}{2\Delta^2}I(0\le x<\Delta)+\left[\frac{x(2\Delta-x)}{2\Delta^2}+\frac{(x-\Delta)(2\Delta-x)}{\Delta^2}\right]I(\Delta\le x<2\Delta)\\&=\frac{x^2}{2\Delta^2}I(0\le x<\Delta)+\left(\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}\right)I(\Delta\le x<2\Delta)\end{aligned}$$

$$\begin{aligned}
B_{5,3}(x) &= \frac{x-\tau_5}{\tau_7-\tau_5} B_{5,2}(x) + \frac{\tau_8-x}{\tau_8-\tau_6} B_{6,2}(x) \\
&= \frac{x-\tau_5}{\tau_7-\tau_5}\left[\frac{x-\Delta}{\Delta} I(\Delta \le x < 2\Delta)\right] + \frac{\tau_8-x}{\tau_8-\tau_6} * 0 \\
&= \frac{x-\Delta}{\Delta}\left[\frac{x-\Delta}{\Delta} I(\Delta \le x < 2\Delta)\right] \\
&= \left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta \le x < 2\Delta)
\end{aligned}$$

$$\begin{aligned}
B_{6,3}(x) &= \frac{x-\tau_6}{\tau_8-\tau_6} B_{6,2}(x) + \frac{\tau_9-x}{\tau_9-\tau_7} B_{7,2}(x) \\
&= \frac{x-\tau_6}{\tau_8-\tau_6} * 0 + \frac{\tau_9-x}{\tau_9-\tau_7} * 0 \\
&= 0
\end{aligned}$$



m=4 K+2M-m=1+8-4=5

$$B_{i,4}(x) = \frac{x-\tau_i}{\tau_{i+3}-\tau_i} B_{i,3}(x) + \frac{\tau_{i+4}-x}{\tau_{i+4}-\tau_{i+1}} B_{i+1,3}(x), i = 1,2,3,4,5$$

$$B_{1,4}(x) = \frac{x-\tau_1}{\tau_4-\tau_1} B_{1,3}(x) + \frac{\tau_5-x}{\tau_5-\tau_2} B_{2,3}(x)$$

$$= \frac{x-\tau_1}{\tau_4-\tau_1} * 0 + \frac{\tau_5-x}{\tau_5-\tau_2}\left[\left(\frac{\Delta-x}{\Delta}\right)^2 I(0 \le x < \Delta)\right]$$

$$= \frac{\Delta-x}{\Delta}\left[\left(\frac{\Delta-x}{\Delta}\right)^2 I(0 \le x < \Delta)\right]$$

$$= \left(\frac{\Delta-x}{\Delta}\right)^3 I(0 \le x < \Delta)$$

$$B_{2,4}(x) = \frac{x-\tau_2}{\tau_5-\tau_2} B_{2,3}(x) + \frac{\tau_6-x}{\tau_6-\tau_3} B_{3,3}(x)$$

$$= \frac{x-\tau_2}{\tau_5-\tau_2}\left[\left(\frac{\Delta-x}{\Delta}\right)^2 I(0 \le x < \Delta)\right]$$

$$+ \frac{\tau_6-x}{\tau_6-\tau_3}\left\{\left(\frac{-3x^2+4x\Delta}{2\Delta^2}\right) I(0 \le x < \Delta) + \frac{(2\Delta-x)^2}{2\Delta^2} I(\Delta \le x < 2\Delta)\right\}$$

$$= \frac{x}{\Delta}\left[\left(\frac{\Delta-x}{\Delta}\right)^2 I(0 \le x < \Delta)\right]$$

$$+ \frac{2\Delta-x}{2\Delta}\left\{\left(\frac{-3x^2+4x\Delta}{2\Delta^2}\right) I(0 \le x < \Delta) + \frac{(2\Delta-x)^2}{2\Delta^2} I(\Delta \le x < 2\Delta)\right\}$$

$$= \left[\frac{x(\Delta-x)^2}{\Delta^3} + \frac{3x^3-10x^2\Delta+8x\Delta^2}{4\Delta^3}\right] I(0 \le x < \Delta) + \frac{(2\Delta-x)^3}{4\Delta^3} I(\Delta \le x < 2\Delta)$$

$$= \left(\frac{7x^3-18x^2\Delta+12x\Delta^2}{4\Delta^3}\right) I(0 \le x < \Delta) + \frac{(2\Delta-x)^3}{4\Delta^3} I(\Delta \le x < 2\Delta)$$

$$B_{3,4}(x) = \frac{x-\tau_3}{\tau_6-\tau_3} B_{3,3}(x) + \frac{\tau_7-x}{\tau_7-\tau_4} B_{4,3}(x)$$

$$= \frac{x-\tau_3}{\tau_6-\tau_3}\left\{\left(\frac{-3x^2+4x\Delta}{2\Delta^2}\right) I(0 \le x < \Delta) + \frac{(2\Delta-x)^2}{2\Delta^2} I(\Delta \le x < 2\Delta)\right\}$$

$$+ \frac{\tau_7-x}{\tau_7-\tau_4}\left\{\frac{x^2}{2\Delta^2} I(0 \le x < \Delta) + \left(\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}\right) I(\Delta \le x < 2\Delta)\right\}$$

$$= \frac{x}{2\Delta}\left\{\left(\frac{-3x^2+4x\Delta}{2\Delta^2}\right) I(0 \le x < \Delta) + \frac{(2\Delta-x)^2}{2\Delta^2} I(\Delta \le x < 2\Delta)\right\}$$

$$+ \frac{2\Delta-x}{2\Delta}\left\{\frac{x^2}{2\Delta^2} I(0 \le x < \Delta) + \left(\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}\right) I(\Delta \le x < 2\Delta)\right\}$$

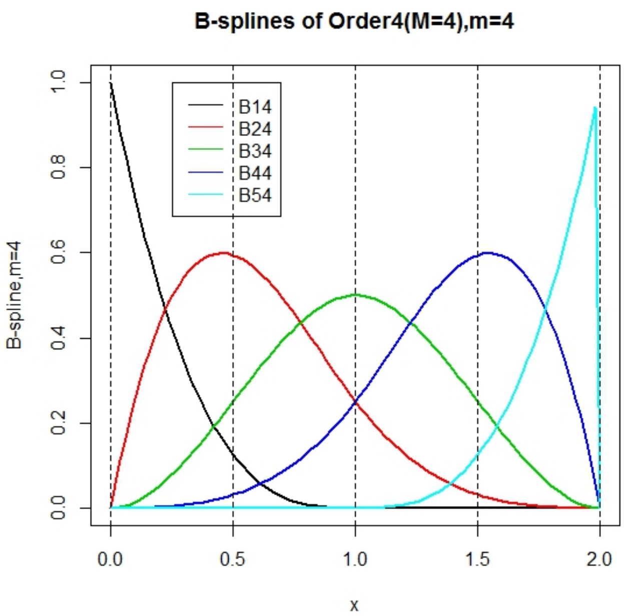
$$= \frac{-4x^3+6x^2\Delta}{4\Delta^3} I(0 \le x < \Delta) + \frac{4x^3-18x^2\Delta+24x\Delta^2-8\Delta^3}{4\Delta^3} I(\Delta \le x < 2\Delta)$$

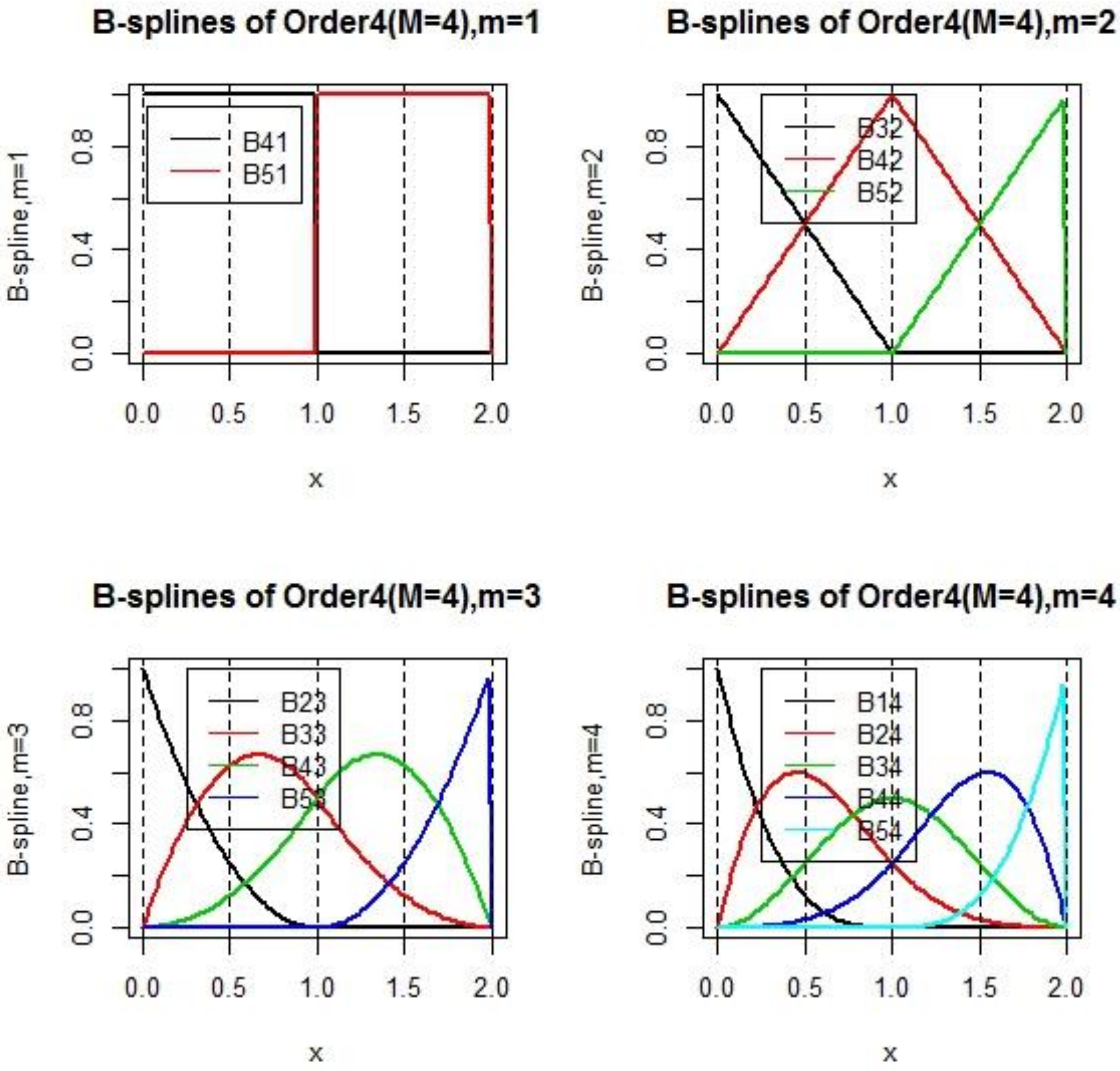
$$= \frac{-2x^3+3x^2\Delta}{2\Delta^3} I(0 \le x < \Delta) + \frac{2x^3-9x^2\Delta+12x\Delta^2-4\Delta^3}{2\Delta^3} I(\Delta \le x < 2\Delta)$$

$$= \frac{x(-2x^2+3x\Delta)}{2\Delta^3} I(0 \le x < \Delta) + \frac{x(2x^2-9x\Delta+12\Delta^2)-4\Delta^3}{2\Delta^3} I(\Delta \le x < 2\Delta)$$

$$
\begin{aligned}
\mathrm{B}_{4,4}(x) &= \frac{x-\tau_4}{\tau_7-\tau_4}\mathrm{B}_{4,3}(x) + \frac{\tau_8-x}{\tau_8-\tau_5}\mathrm{B}_{5,3}(x) \\
&= \frac{x-\tau_4}{\tau_7-\tau_4}\left\{\frac{x^2}{2\Delta^2}I(0\le x<\Delta) + \left(\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}\right)I(\Delta\le x<2\Delta)\right\} \\
&\quad + \frac{\tau_8-x}{\tau_8-\tau_5}\left[\left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta\le x<2\Delta)\right] \\
&= \frac{x}{2\Delta}\left\{\frac{x^2}{2\Delta^2}I(0\le x<\Delta) + \left(\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}\right)I(\Delta\le x<2\Delta)\right\} \\
&\quad + \frac{2\Delta-x}{\Delta}\left[\left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta\le x<2\Delta)\right] \\
&= \frac{x^3}{4\Delta^3}I(0\le x<\Delta) + \left(\frac{-3x^3+8x^2\Delta-4x\Delta^2}{4\Delta^3} + \frac{-x^3+4x^2\Delta-5x\Delta^2+2\Delta^3}{\Delta^3}\right)I(\Delta\le x<2\Delta) \\
&= \frac{x^3}{4\Delta^3}I(0\le x<\Delta) + \left(\frac{-7x^3+24x^2\Delta-24x\Delta^2+8\Delta^3}{4\Delta^3}\right)I(\Delta\le x<2\Delta)
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{B}_{5,4}(x) &= \frac{x-\tau_5}{\tau_8-\tau_5}\mathrm{B}_{5,3}(x) + \frac{\tau_9-x}{\tau_9-\tau_6}\mathrm{B}_{6,3}(x) \\
&= \frac{x-\tau_5}{\tau_8-\tau_5}\left[\left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta\le x<2\Delta)\right] + \frac{\tau_9-x}{\tau_9-\tau_6} * 0 \\
&= \frac{x-\Delta}{\Delta}\left[\left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta\le x<2\Delta)\right] \\
&= \left(\frac{x-\Delta}{\Delta}\right)^3 I(\Delta\le x<2\Delta)
\end{aligned}
$$





Comments

As we seen above, cubic B-spline bases (order4) with knots which are equally spaced have much more smooth trend when m is increasing.

2. Draw a figure for order1~4 with $\xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta$ (like Figure 5.20)

- **Order1(M=1), m=1**

Let $\begin{cases} K = 1 \;\; M = 1 \\ \xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta \end{cases}$

K+2M=1+2=3

Augmented knots are

$\tau_1 \le \tau_2 \le \tau_3$ s.t.

1) $\tau_1 = \xi_0 = 0$

2) $\tau_2 = \xi_1 = \Delta$

3) $\tau_3 = \xi_2 = 2\Delta$

- Note: In order to be equally spaced $(\Delta = \xi_1 - \xi_0 = \xi_2 - \xi_1)$, I use $\Delta = 1$ in R program.

K+2M-1=1+2-1=2

$B_{i,1}(x) = I(\tau_i \le x < \tau_{i+1}), i = 1,2$

$B_{1,1}(x) = I(0 \le x < \Delta)$

$B_{2,1}(x) = I(\Delta \le x < 2\Delta)$

Order2(M=2), m=1

Let $\begin{cases} K = 1 \;\; M = 2 \\ \xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta \end{cases}$

K+2M=1+4=5

Augmented knots are

$\tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le \tau_5$ s.t.

1) $\tau_1 = \tau_2 = \xi_0 = 0$
2) $\tau_3 = \xi_1 = \Delta$
3) $\tau_4 = \tau_5 = \xi_2 = 2\Delta$

- Note: In order to be equally spaced $(\Delta = \xi_1 - \xi_0 = \xi_2 - \xi_1)$, I use $\Delta = 1$ in R program.

K+2M-1=1+4-1=4

$\mathrm{B}_{i,1}(x) = I(\tau_i \le x < \tau_{i+1}), i = 1,2,3,4$

$\mathrm{B}_{1,1}(x) = 0 \quad \because \tau_1 = \tau_2$

$\mathrm{B}_{2,1}(x) = I(0 \le x < \Delta)$

$\mathrm{B}_{3,1}(x) = I(\Delta \le x < 2\Delta)$

$\mathrm{B}_{4,1}(x) = 0 \quad \because \tau_4 = \tau_5$

Order2(M=2), m=2

K+2M-m=1+4-2=3

$$\mathrm{B}_{i,2}(x) = \frac{x - \tau_i}{\tau_{i+1} - \tau_i}\mathrm{B}_{i,1}(x) + \frac{\tau_{i+2} - x}{\tau_{i+2} - \tau_{i+1}}\mathrm{B}_{i+1,1}(x), i = 1,2,3$$

$$\begin{aligned}\mathrm{B}_{1,2}(x) &= \frac{x - \tau_1}{\tau_2 - \tau_1}\mathrm{B}_{1,1}(x) + \frac{\tau_3 - x}{\tau_3 - \tau_2}\mathrm{B}_{2,1}(x) \\ &= \frac{x - \tau_1}{\tau_2 - \tau_1} * 0 + \frac{\tau_3 - x}{\tau_3 - \tau_2} I(0 \le x < \Delta) \\ &= \frac{\Delta - x}{\Delta} I(0 \le x < \Delta)\end{aligned}$$

$$\begin{aligned}\mathrm{B}_{2,2}(x) &= \frac{x - \tau_2}{\tau_3 - \tau_2}\mathrm{B}_{2,1}(x) + \frac{\tau_4 - x}{\tau_4 - \tau_3}\mathrm{B}_{3,1}(x) \\ &= \frac{x - \tau_2}{\tau_3 - \tau_2} I(0 \le x < \Delta) + \frac{\tau_4 - x}{\tau_4 - \tau_3} I(\Delta \le x < 2\Delta) \\ &= \frac{x}{\Delta} I(0 \le x < \Delta) + \frac{2\Delta - x}{\Delta} I(\Delta \le x < 2\Delta)\end{aligned}$$

$$\begin{aligned}\mathrm{B}_{3,2}(x) &= \frac{x - \tau_3}{\tau_4 - \tau_3}\mathrm{B}_{3,1}(x) + \frac{\tau_5 - x}{\tau_5 - \tau_4}\mathrm{B}_{4,1}(x) \\ &= \frac{x - \tau_3}{\tau_4 - \tau_3} I(\Delta \le x < 2\Delta) + \frac{\tau_5 - x}{\tau_5 - \tau_4} * 0 \\ &= \frac{x - \Delta}{\Delta} I(\Delta \le x < 2\Delta)\end{aligned}$$

Order3(M=3), m=1

Let $\begin{cases} K = 1 \quad M = 3 \\ \xi_0 = 0, \xi_1 = \Delta, \xi_2 = 2\Delta \end{cases}$

K+2M=1+6=7

Augmented knots are

$\tau_1 \le \tau_2 \le \tau_3 \le \tau_4 \le \tau_5 \le \tau_6 \le \tau_7$ s.t.

1) $\tau_1 = \tau_2 = \tau_3 = \xi_0 = 0$
2) $\tau_4 = \xi_1 = \Delta$
3) $\tau_5 = \tau_6 = \tau_7 = \xi_2 = 2\Delta$

- Note: In order to be equally spaced $(\Delta = \xi_1 - \xi_0 = \xi_2 - \xi_1)$, I use $\Delta = 1$ in R program.

K+2M-1=1+6-1=6

$\mathrm{B}_{i,1}(x) = I(\tau_i \le x < \tau_{i+1}), i = 1,2,3,4,5,6$

$\mathrm{B}_{1,1}(x) = 0 \quad \because \tau_1 = \tau_2$

$\mathrm{B}_{2,1}(x) = 0 \quad \because \tau_2 = \tau_3$

$\mathrm{B}_{3,1}(x) = I(0 \le x < \Delta)$

$\mathrm{B}_{4,1}(x) = I(\Delta \le x < 2\Delta)$

$\mathrm{B}_{5,1}(x) = 0 \quad \because \tau_5 = \tau_6$

$\mathrm{B}_{6,1}(x) = 0 \quad \because \tau_6 = \tau_7$

Order3(M=3), m=2

K+2M-m=1+6-2=5

$$\mathrm{B}_{i,2}(x) = \frac{x - \tau_i}{\tau_{i+1} - \tau_i}\mathrm{B}_{i,1}(x) + \frac{\tau_{i+2} - x}{\tau_{i+2} - \tau_{i+1}}\mathrm{B}_{i+1,1}(x), i = 1,2,3,4,5$$

$$\mathrm{B}_{1,2}(x) = \frac{x - \tau_1}{\tau_2 - \tau_1}\mathrm{B}_{1,1}(x) + \frac{\tau_3 - x}{\tau_3 - \tau_2}\mathrm{B}_{2,1}(x) = \frac{x - \tau_1}{\tau_2 - \tau_1} * 0 + \frac{\tau_3 - x}{\tau_3 - \tau_2} * 0 = 0$$

$$\mathrm{B}_{2,2}(x) = \frac{x - \tau_2}{\tau_3 - \tau_2}\mathrm{B}_{2,1}(x) + \frac{\tau_4 - x}{\tau_4 - \tau_3}\mathrm{B}_{3,1}(x) = \frac{x - \tau_2}{\tau_3 - \tau_2} * 0 + \frac{\tau_4 - x}{\tau_4 - \tau_3}I(0 \le x < \Delta) = \frac{\Delta - x}{\Delta}I(0 \le x < \Delta)$$

$$\mathrm{B}_{3,2}(x) = \frac{x - \tau_3}{\tau_4 - \tau_3}\mathrm{B}_{3,1}(x) + \frac{\tau_5 - x}{\tau_5 - \tau_4}\mathrm{B}_{4,1}(x) = \frac{x - \tau_3}{\tau_4 - \tau_3}I(0 \le x < \Delta) + \frac{\tau_5 - x}{\tau_5 - \tau_4}I(\Delta \le x < 2\Delta)$$

$$= \frac{x}{\Delta}I(0 \le x < \Delta) + \frac{2\Delta - x}{\Delta}I(\Delta \le x < 2\Delta)$$

$$\mathrm{B}_{4,2}(x) = \frac{x - \tau_4}{\tau_5 - \tau_4}\mathrm{B}_{4,1}(x) + \frac{\tau_6 - x}{\tau_6 - \tau_5}\mathrm{B}_{5,1}(x) = \frac{x - \tau_4}{\tau_5 - \tau_4}I(\Delta \le x < 2\Delta) + \frac{\tau_6 - x}{\tau_6 - \tau_5} * 0$$

$$= \frac{x - \Delta}{\Delta}I(\Delta \le x < 2\Delta)$$

$$\mathrm{B}_{5,2}(x) = \frac{x - \tau_5}{\tau_6 - \tau_5}\mathrm{B}_{5,1}(x) + \frac{\tau_7 - x}{\tau_7 - \tau_6}\mathrm{B}_{6,1}(x) = \frac{x - \tau_5}{\tau_6 - \tau_5} * 0 + \frac{\tau_7 - x}{\tau_7 - \tau_6} * 0 = 0$$

- **Order3(M=3), m=3**

K+2M-m=1+6-3=4

$$\mathrm{B}_{i,3}(x)=\frac{x-\tau_i}{\tau_{i+2}-\tau_i}\mathrm{B}_{i,2}(x)+\frac{\tau_{i+3}-x}{\tau_{i+3}-\tau_{i+1}}\mathrm{B}_{i+1,2}(x), i=1,2,3,4$$

$$\mathrm{B}_{1,3}(x)=\frac{x-\tau_1}{\tau_3-\tau_1}\mathrm{B}_{1,2}(x)+\frac{\tau_4-x}{\tau_4-\tau_2}\mathrm{B}_{2,2}(x)=\frac{x-\tau_1}{\tau_3-\tau_1}*0+\frac{\tau_4-x}{\tau_4-\tau_2}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]$$

$$=\frac{\Delta-x}{\Delta}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]$$

$$=\left(\frac{\Delta-x}{\Delta}\right)^2 I(0\le x<\Delta)$$

$$\mathrm{B}_{2,3}(x)=\frac{x-\tau_2}{\tau_4-\tau_2}\mathrm{B}_{2,2}(x)+\frac{\tau_5-x}{\tau_5-\tau_3}\mathrm{B}_{3,2}(x)$$

$$=\frac{x-\tau_2}{\tau_4-\tau_2}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]+\frac{\tau_5-x}{\tau_5-\tau_3}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]$$

$$=\frac{x}{\Delta}\left[\frac{\Delta-x}{\Delta}I(0\le x<\Delta)\right]+\frac{2\Delta-x}{2\Delta}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]$$

$$=\left[\frac{x(\Delta-x)}{\Delta^2}+\frac{x(2\Delta-x)}{2\Delta^2}\right]I(0\le x<\Delta)+\frac{(2\Delta-x)^2}{2\Delta^2}I(\Delta\le x<2\Delta)$$

$$=\frac{-3x^2+4x\Delta}{2\Delta^2}I(0\le x<\Delta)+\frac{(2\Delta-x)^2}{2\Delta^2}I(\Delta\le x<2\Delta)$$

$$\mathrm{B}_{3,3}(x)=\frac{x-\tau_3}{\tau_5-\tau_3}\mathrm{B}_{3,2}(x)+\frac{\tau_6-x}{\tau_6-\tau_4}\mathrm{B}_{4,2}(x)$$

$$=\frac{x-\tau_3}{\tau_5-\tau_3}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]+\frac{\tau_6-x}{\tau_6-\tau_4}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]$$

$$=\frac{x}{2\Delta}\left[\frac{x}{\Delta}I(0\le x<\Delta)+\frac{2\Delta-x}{\Delta}I(\Delta\le x<2\Delta)\right]+\frac{2\Delta-x}{\Delta}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]$$

$$=\frac{x^2}{2\Delta^2}I(0\le x<\Delta)+\left[\frac{x(2\Delta-x)}{2\Delta^2}+\frac{(x-\Delta)(2\Delta-x)}{\Delta^2}\right]I(\Delta\le x<2\Delta)$$

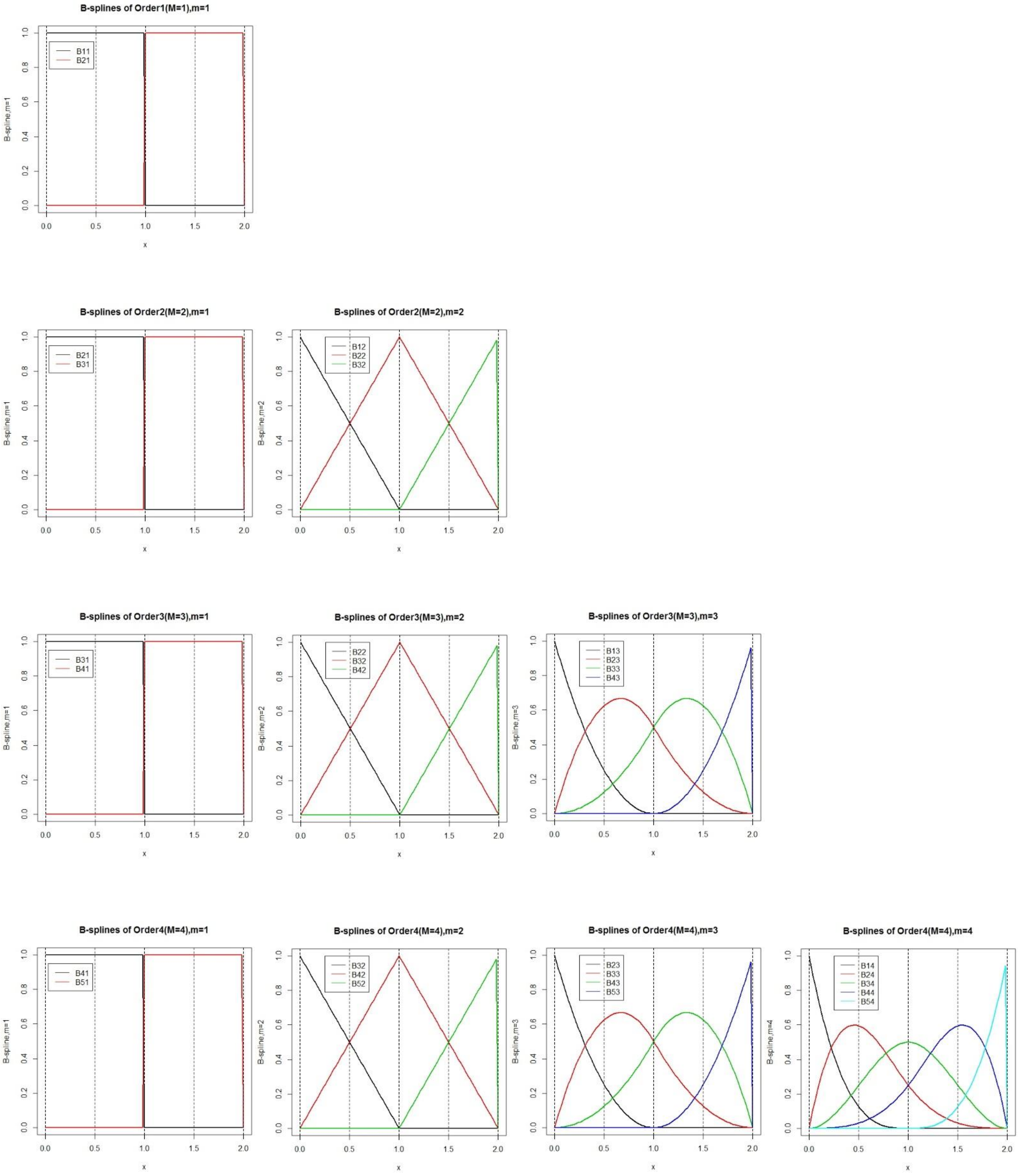
$$=\frac{x^2}{2\Delta^2}I(0\le x<\Delta)+\frac{-3x^2+8x\Delta-4\Delta^2}{2\Delta^2}I(\Delta\le x<2\Delta)$$

$$\mathrm{B}_{4,3}(x)=\frac{x-\tau_4}{\tau_6-\tau_4}\mathrm{B}_{4,2}(x)+\frac{\tau_7-x}{\tau_7-\tau_5}\mathrm{B}_{5,2}(x)=\frac{x-\tau_4}{\tau_6-\tau_4}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]+\frac{\tau_7-x}{\tau_7-\tau_5}*0$$

$$=\frac{x-\Delta}{\Delta}\left[\frac{x-\Delta}{\Delta}I(\Delta\le x<2\Delta)\right]$$

$$=\left(\frac{x-\Delta}{\Delta}\right)^2 I(\Delta\le x<2\Delta)$$

Figures



Comments

After a series of complicated computations of splines, I plot all B-splines figures under different conditions. By the figures, I obviously find that ❶when m=1, B-splines patterns of order1-4 are the same, only differ from the *i*th B-spline basis ; ❷when m=2, B-splines patterns of order2-4 are the same, only differ from the *i*th B-spline basis ; ❸when m=3, B-splines patterns of order3-4 are the same, only differ from the *i*th B-spline basis ; ❹when order is increasing, the pattern becomes more smooth.

Appendix

R Codes

```
###20161122_HW4###
##1:M=4##
par(mfrow=c(2,2))
#m=1#
rm(list=ls(all=TRUE))
delta     = 1
B11       = 0
B21       = 0
B31       = 0
B41       = function(x){(x>=0&x<delta)}
B51       = function(x){(x>=delta&x<2*delta)}
B61       = 0
B71       = 0
B81       = 0
curve(B41,xlim=c(0,2),main="B-splines of Order4(M=4),m=1",ylab="B-
spline,m=1",font.main=2,col=1,lty=1,lwd=2)
curve(B51,xlim=c(0,2),main="B-splines of Order4(M=4),m=1",ylab="B-
spline,m=1",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.03,0.95,legend=c("B41","B51"),lty=1,col=c(1:2))
#m=2#
rm(list=ls(all=TRUE))
delta     = 1
B12       = 0
B22       = 0
B32       = function(x){((delta-x)/delta)*(x>=0&x<delta)}
B42       = function(x){(x/delta)*(x>=0&x<delta)+((2*delta-x)/delta)*(x>=delta&x<2*delta)}
B52       = function(x){((x-delta)/delta)*(x>=delta&x<2*delta)}
B62       = 0
B72       = 0
curve(B32,xlim=c(0,2),main="B-splines of Order4(M=4),m=2",ylab="B-
spline,m=2",font.main=2,col=1,lty=1,lwd=2)
curve(B42,xlim=c(0,2),main="B-splines of Order4(M=4),m=2",ylab="B-
spline,m=2",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
```

```
curve(B52,xlim=c(0,2),main="B-splines of Order4(M=4),m=2",ylab="B-
spline,m=2",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B32","B42","B52"),lty=1,col=c(1:3))
#m=3#
rm(list=ls(all=TRUE))
delta       = 1
B13         = 0
B23         = function(x){((delta-x)/delta)^2*(x>=0&x<delta)}
B33         = function(x){((-3*(x^2)+4*x*delta)/(2*(delta^2)))*(x>=0&x<delta)+((2*delta-
x)^2/(2*(delta^2)))*(x>=delta&x<2*delta)}
B43         = function(x){(x^2/(2*(delta^2)))*(x>=0&x<delta)+((-3*(x^2)+8*x*delta-
4*(delta^2))/(2*(delta^2)))*(x>=delta&x<2*delta)}
B53         = function(x){((x-delta)/delta)^2*(x>=delta&x<2*delta)}
B63         = 0
curve(B23,xlim=c(0,2),main="B-splines of Order4(M=4),m=3",ylab="B-
spline,m=3",font.main=2,col=1,lty=1,lwd=2)
curve(B33,xlim=c(0,2),main="B-splines of Order4(M=4),m=3",ylab="B-
spline,m=3",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
curve(B43,xlim=c(0,2),main="B-splines of Order4(M=4),m=3",ylab="B-
spline,m=3",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
curve(B53,xlim=c(0,2),main="B-splines of Order4(M=4),m=3",ylab="B-
spline,m=3",font.main=2,col=4,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B23","B33","B43","B53"),lty=1,col=c(1:4))
#m=4#
rm(list=ls(all=TRUE))
delta       = 1
B14         = function(x){((delta-x)/delta)^3*(x>=0&x<delta)}
B24         = function(x){((7*x^3-18*x^2*delta+12*x*delta^2)/(4*delta^3))*(x>=0&x<delta)+((2*delta-
x)^3/(4*(delta^3)))*(x>=delta&x<2*delta)}
```

```
B34         = function(x){((-2*x^3+3*x^2*delta)/(2*delta^3))*(x>=0&x<delta)+((2*x^3-
9*x^2*delta+12*x*delta^2-4*delta^3)/(2*delta^3))*(x>=delta&x<2*delta)}
B44         = function(x){(x^3/(4*delta^3))*(x>=0&x<delta)+((-7*x^3+24*x^2*delta-
24*x*delta^2+8*delta^3)/(4*delta^3))*(x>=delta&x<2*delta)}
B54         = function(x){((x-delta)/delta)^3*(x>=delta&x<2*delta)}
curve(B14,xlim=c(0,2),main="B-splines of Order4(M=4),m=4",ylab="B-
spline,m=4",font.main=2,col=1,lty=1,lwd=2)
curve(B24,xlim=c(0,2),main="B-splines of Order4(M=4),m=4",ylab="B-
spline,m=4",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
curve(B34,xlim=c(0,2),main="B-splines of Order4(M=4),m=4",ylab="B-
spline,m=4",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
curve(B44,xlim=c(0,2),main="B-splines of Order4(M=4),m=4",ylab="B-
spline,m=4",font.main=2,col=4,lty=1,lwd=2,add=TRUE)
curve(B54,xlim=c(0,2),main="B-splines of Order4(M=4),m=4",ylab="B-
spline,m=4",font.main=2,col=5,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B14","B24","B34","B44","B54"),lty=1,col=c(1:5))
##############################################################################################
##2##
#2_1:M=1,m=1#
rm(list=ls(all=TRUE))
delta       = 1
B11         = function(x){(x>=0&x<delta)}
B21         = function(x){(x>=delta&x<2*delta)}
curve(B11,xlim=c(0,2),main="B-splines of Order1(M=1),m=1",ylab="B-
spline,m=1",font.main=2,col=1,lty=1,lwd=2)
curve(B21,xlim=c(0,2),main="B-splines of Order1(M=1),m=1",ylab="B-
spline,m=1",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.03,0.95,legend=c("B11","B21"),lty=1,col=c(1:2))
#2_2_1:M=2,m=1#
rm(list=ls(all=TRUE))
```

```
delta       = 1
B11         = 0
B21         = function(x){(x>=0&x<delta)}
B31         = function(x){(x>=delta&x<2*delta)}
B41         = 0
curve(B21,xlim=c(0,2),main="B-splines of Order2(M=2),m=1",ylab="B-
spline,m=1",font.main=2,col=1,lty=1,lwd=2)
curve(B31,xlim=c(0,2),main="B-splines of Order2(M=2),m=1",ylab="B-
spline,m=1",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.03,0.95,legend=c("B21","B31"),lty=1,col=c(1:2))
#2_2_2:M=2,m=2#
rm(list=ls(all=TRUE))
delta       = 1
B12         = function(x){((delta-x)/delta)*(x>=0&x<delta)}
B22         = function(x){(x/delta)*(x>=0&x<delta)+((2*delta-x)/delta)*(x>=delta&x<2*delta)}
B32         = function(x){((x-delta)/delta)*(x>=delta&x<2*delta)}
curve(B12,xlim=c(0,2),main="B-splines of Order2(M=2),m=2",ylab="B-
spline,m=2",font.main=2,col=1,lty=1,lwd=2)
curve(B22,xlim=c(0,2),main="B-splines of Order2(M=2),m=2",ylab="B-
spline,m=2",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
curve(B32,xlim=c(0,2),main="B-splines of Order2(M=2),m=2",ylab="B-
spline,m=2",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B12","B22","B32"),lty=1,col=c(1:3))
#2_3_1:M=3,m=1#
rm(list=ls(all=TRUE))
delta       = 1
B11         = 0
B21         = 0
B31         = function(x){(x>=0&x<delta)}
B41         = function(x){(x>=delta&x<2*delta)}
```

```
B51 	= 0
B61 	= 0
curve(B31,xlim=c(0,2),main="B-splines of Order3(M=3),m=1",ylab="B-
spline,m=1",font.main=2,col=1,lty=1,lwd=2)
curve(B41,xlim=c(0,2),main="B-splines of Order3(M=3),m=1",ylab="B-
spline,m=1",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.03,0.95,legend=c("B31","B41"),lty=1,col=c(1:2))
#2_3_2:M=3,m=2#
rm(list=ls(all=TRUE))
delta 	= 1
B12 	= 0
B22 	= function(x){((delta-x)/delta)*(x>=0&x<delta)}
B32 	= function(x){(x/delta)*(x>=0&x<delta)+((2*delta-x)/delta)*(x>=delta&x<2*delta)}
B42 	= function(x){((x-delta)/delta)*(x>=delta&x<2*delta)}
B52 	= 0
curve(B22,xlim=c(0,2),main="B-splines of Order3(M=3),m=2",ylab="B-
spline,m=2",font.main=2,col=1,lty=1,lwd=2)
curve(B32,xlim=c(0,2),main="B-splines of Order3(M=3),m=2",ylab="B-
spline,m=2",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
curve(B42,xlim=c(0,2),main="B-splines of Order3(M=3),m=2",ylab="B-
spline,m=2",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B22","B32","B42"),lty=1,col=c(1:3))
#2_3_3:M=3,m=3#
rm(list=ls(all=TRUE))
delta 	= 1
B13 	= function(x){((delta-x)/delta)^2*(x>=0&x<delta)}
B23 	= function(x){((-3*(x^2)+4*x*delta)/(2*(delta^2)))*(x>=0&x<delta)+((2*delta-
x)^2/(2*(delta^2)))*(x>=delta&x<2*delta)}
B33 	= function(x){(x^2/(2*(delta^2)))*(x>=0&x<delta)+((-3*(x^2)+8*x*delta-
4*(delta^2))/(2*(delta^2)))*(x>=delta&x<2*delta)}
```

```
B43          = function(x){((x-delta)/delta)^2*(x>=delta&x<2*delta)}
curve(B13,xlim=c(0,2),main="B-splines of Order3(M=3),m=3",ylab="B-
spline,m=3",font.main=2,col=1,lty=1,lwd=2)
curve(B23,xlim=c(0,2),main="B-splines of Order3(M=3),m=3",ylab="B-
spline,m=3",font.main=2,col=2,lty=1,lwd=2,add=TRUE)
curve(B33,xlim=c(0,2),main="B-splines of Order3(M=3),m=3",ylab="B-
spline,m=3",font.main=2,col=3,lty=1,lwd=2,add=TRUE)
curve(B43,xlim=c(0,2),main="B-splines of Order3(M=3),m=3",ylab="B-
spline,m=3",font.main=2,col=4,lty=1,lwd=2,add=TRUE)
abline(v=0,lty=2)
abline(v=0.5,lty=2)
abline(v=1,lty=2)
abline(v=1.5,lty=2)
abline(v=2,lty=2)
#locator()
legend(0.25,1,legend=c("B13","B23","B33","B43"),lty=1,col=c(1:4))
```