A robust change point estimator for binomial CUSUM control charts

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- Introduction

- Background
  1. Binomial CUSUM control chart
  2. Maximum likelihood estimator
  3. Page’s last zero estimator

- Method

  Combine CUSUM estimator and MLE

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- Simulations

  Design closely follows those of Perry and Pignatiello (2005)

- Data analysis

  Jewelry manufacturing data by Burr (1979)

- Conclusion

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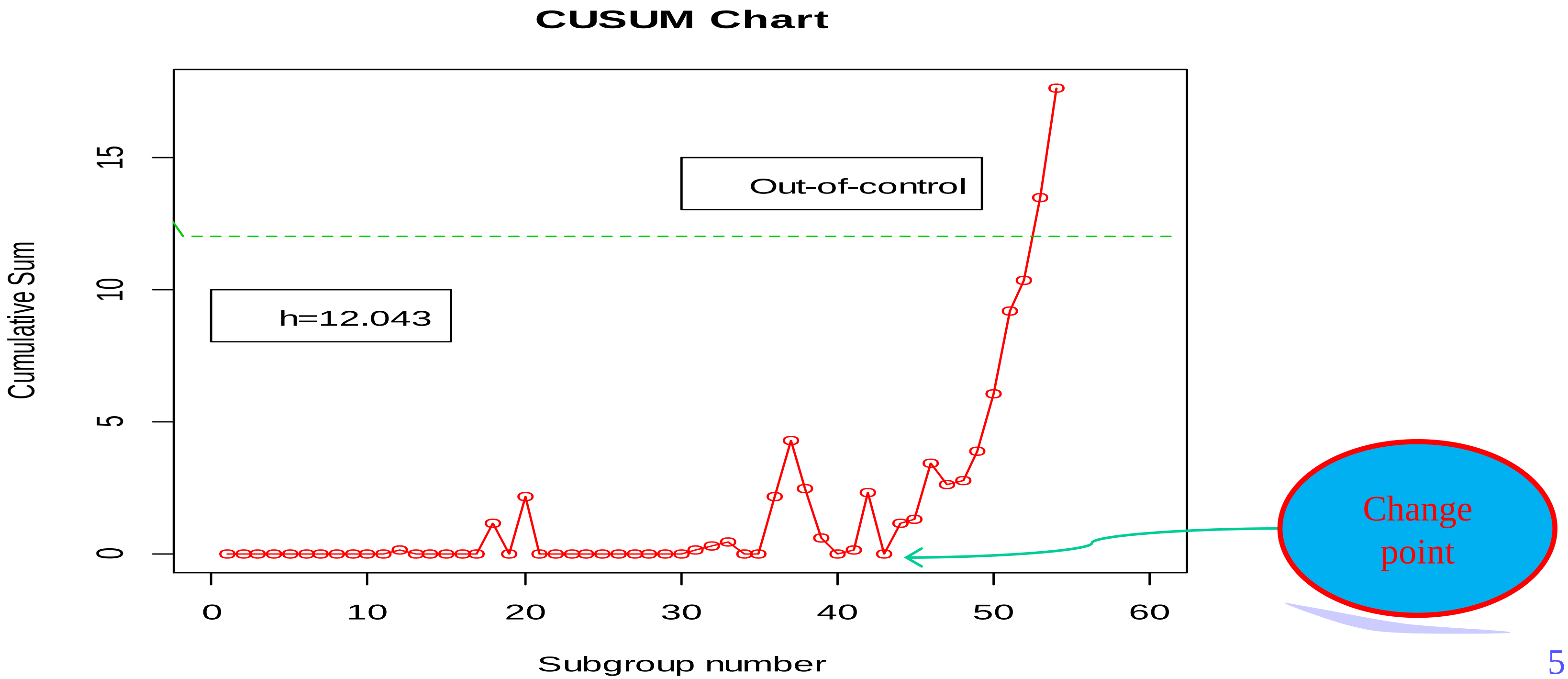
Introduction

- What is the change point?

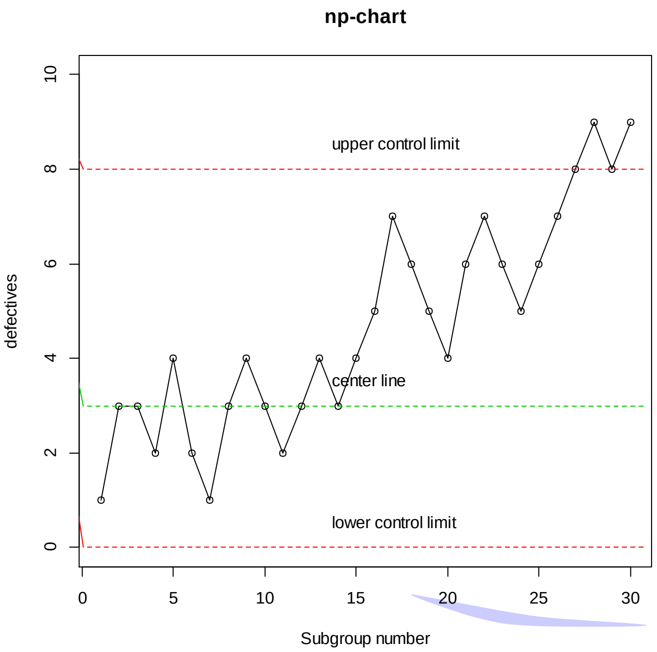
- We consider observations come from binomial distribution with the same fraction nonconforming p .



Introduction

In SPC, the *np*-chart is most famous charts used to monitor the number of nonconforming items for industrial manufactures.

When sample size is large and defective rate is not too small, the *np*-chart works well.



Introduction

Page (1954, 1955) first suggested the CUSUM chart to estimate the change point.

The binomial CUSUM control chart is a good alternative when small changes are important.

Introduction

Samuel and Pignatiello (2001) proposed maximum likelihood estimator (MLE) for the process change point using the step change Likelihood function for a binomial random variable.

Perry and Pignatiello (2005) shows that the performance of the MLE is often better than Page's last zero estimator.

Introduction

The MLE method outperforms CUSUM method when magnitudes of change is large.

In order to construct more robust in different parameter setting, this thesis combines CUSUM estimator and MLE. Furthermore, compares the new method with two estimators.

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Binomial CUSUM control chart

$$X_1, X_2, \ldots, X_\tau \overset{i.i.d}{\sim} Bin(n, p_0), \quad X_{\tau+1}, X_{\tau+2}, \ldots, X_T \overset{i.i.d}{\sim} Bin(n, p_a^{True})$$

$\Theta = \{ (\tau, p_a^{True}) \mid \tau \in \{1, 2, \ldots, T\}, p_0 \le p_a^{True} \le 1 \}$ for a fixed value

$T \in \{1, 2, \cdots\}$.

$$P(X_i = x) = \begin{cases} \binom{n}{x} p_0^{\,x} (1 - p_0)^{n-x}, & \text{if} \quad i = 1, 2, \cdots, \tau, \\ \binom{n}{x} (p_a^{True})^x (1 - p_a^{True})^{n-x}, & \text{if} \quad i = \tau+1, \tau+2, \cdots, T. \end{cases}$$

p_0 : in-control fraction nonconforming.

p_a^{True}: out-of-control fraction nonconforming.

τ : the true change point .

Binomial CUSUM control chart

$$S_0 = 0 \text{ and } S_i = \max\{\, 0,\, X_i - nk + S_{i-1} \,\}, \qquad i = 1, 2, \ldots$$

where

$$k = \frac{-\ln\left(\dfrac{1 - p_a^{as}}{1 - p_0} \right)}{\ln\left\{ \dfrac{p_a^{as}(1 - p_0)}{p_0(1 - p_a^{as})} \right\}},$$

k : The reference value.

p_a^{as} : out-of-control fraction nonconforming for which to design the CUSUM chart.

Binomial CUSUM control chart

When S_i exceeds decision interval $h > 0$ the chart signals that an increase in the process fraction nonconforming has occurred.

Maximum likelihood estimator (MLE)

$$X_1, X_2, \ldots, X_\tau \overset{i.i.d}{\sim} Bin(n, p_0), \quad X_{\tau+1}, X_{\tau+2}, \ldots, X_T \overset{i.i.d}{\sim} Bin(n, p_a^{True})$$

$\Theta = \{ (\tau, p_a^{True}) \mid \tau \in \{1, 2, \ldots, T\}, p_0 \le p_a^{True} \le 1 \}$ for fixed value

$T \in \{1, 2, \cdots\}$.

The likelihood function is given by

$$L(\tau, p_a^{True} \mid p_0, \mathrm{X}) = \prod_{i=1}^{\tau} \binom{n}{x_i} p_0^{\,x_i} (1 - p_0)^{n - x_i} \prod_{i=\tau+1}^{T} \binom{n}{x_i} (p_a^{True})^{x_i} (1 - p_a^{True})^{n - x_i}$$

where $\mathrm{X} = (x_1, x_2, \cdots, x_T)$.

Maximum likelihood estimator (MLE)

The log-likelihood is

$$\log L(\tau, p_a^{True} \mid p_0, \mathrm{X}) = C + \log(p_0)\sum_{i=1}^{\tau} x_i + \log(1-p_0)\sum_{i=1}^{\tau}(n-x_i)$$
$$+\log(p_a^{True})\sum_{i=\tau+1}^{T} x_i + \log(1-p_a^{True})\sum_{i=\tau+1}^{T}(n-x_i),$$

where C is a constant. We can rewrite $\log L(\tau, p_a^{True} \mid p_0, \mathrm{X})$ as

$$\log L(\tau, p_a^{True} \mid p_0, \mathrm{X}) = C^* + \log\left(\frac{p_a^{True}}{p_0}\right)\sum_{i=\tau+1}^{T} x_i + \log\left(\frac{1-p_a^{True}}{1-p_0}\right)\sum_{i=\tau+1}^{T}(n-x_i),$$

where $C^* = C + \log(p_0)\sum_{i=1}^{T} x_i + \log(1-p_0)\sum_{i=1}^{T}(n-x_i)$ is a constant.

Maximum likelihood estimator (MLE)

If τ is given, the log-likelihood equation becomes

$$\frac{\partial}{\partial p_a^{True}} \log L(\tau, p_a^{True} \mid p_0, \mathrm{X}) = \frac{\sum_{i=\tau+1}^{T} x_i}{p_a^{True}} - \frac{\sum_{i=\tau+1}^{T} (n - x_i)}{1 - p_a^{True}},$$

$$\frac{\sum_{i=\tau+1}^{T} x_i}{p_a^{True}} = \frac{\sum_{i=\tau+1}^{T} (n - x_i)}{1 - p_a^{True}} \quad \Rightarrow \quad \hat{p}_a^{True}(\tau) = \frac{\sum_{i=\tau+1}^{T} x_i}{\sum_{i=\tau+1}^{T} n}.$$

Maximum likelihood estimator (MLE)

$$\frac{\partial^2}{\partial p_a^{True^2}} \log L(\tau, p_a \mid p_0, \mathrm{X}) = -\frac{\sum_{i=\tau+1}^{T} x_i}{p_a^{True^2}} - \frac{\sum_{i=\tau+1}^{T}(n - x_i)}{(1 - p_a^{True})^2} < 0.$$

Putting $\hat{p}_a^{True}(\tau)$ into $\log L(\tau, p_a^{True} \mid p_0, \mathrm{X})$, we have a profile log-likelihood for τ as

$$\log L(\tau, \hat{p}_a^{True}(\tau) \mid p_0, \mathrm{X}) = C^* + \log\left(\frac{\hat{p}_a^{True}(\tau)}{p_0}\right)\sum_{i=\tau+1}^{T} x_i + \log\left(\frac{1 - \hat{p}_a^{True}(\tau)}{1 - p_0}\right)\sum_{i=\tau+1}^{T}(n - x_i).$$

Maximum likelihood estimator (MLE)

Therefore, the change point estimator $\hat{\tau}_{MLE}$ is

$$\hat{\tau}_{MLE} = \underset{\tau \in \{1,2,\cdots,T\}}{\arg\max} \left[\log\left\{ \frac{\hat{p}_a^{True}(\tau)}{p_0} \right\} \sum_{i=\tau+1}^{T} x_i + \log\left\{ \frac{1-\hat{p}_a^{True}(\tau)}{1-p_0} \right\} \sum_{i=\tau+1}^{T} (n - x_i) \right].$$

Page's last zero estimator

Under CUSUM control chart we have cumulative sum

$$S_0 = 0 \quad \text{and } S_i = \max\{ 0, X_i - nk + S_{i-1} \}, \quad i = 1, 2, \ldots$$

An estimate of the change point is given by $\hat{\tau}_{CUSUM} = \max\{ i : S_i = 0 \}$.

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Proposed method

We propose a new change point estimator that combines $\hat{\tau}_{MLE}$ and $\hat{\tau}_{\text{CUSUM}}$.

$$\hat{\tau}_{NEW}(w) = w\hat{\tau}_{CUSUM} + (1-w)\hat{\tau}_{MLE}, \qquad 0 \le w \le 1,$$

$$w \equiv w(p_a^{True} \mid p_a^{as}) = \begin{cases} \left(\dfrac{p_a^{True} - p_0}{p_a^{as} - p_0} \right)^{\left(\frac{p_a^{True}}{p_0} \right)}, & \text{if } p_a^{True} - p_0 \le p_a^{as} - p_0, \\ \left(\dfrac{p_a^{as} - p_0}{p_a^{True} - p_0} \right)^{\left(\frac{p_a^{True}}{p_0} \right)}, & \text{if } p_a^{True} - p_0 > p_a^{as} - p_0. \end{cases}$$

Proposed method

First, we estimate unknown p_a^{True} by MLE.

$$\hat{p}_a(\hat{\tau}_{MLE}) = \frac{\sum_{i=\hat{\tau}_{MLE}+1}^{T} x_i}{\sum_{i=\hat{\tau}_{MLE}+1}^{T} n},$$

The estimator of weight function as

$$\hat{w} \equiv w(\hat{p}_a(\hat{\tau}_{MLE}) \mid p_a^{as}) = \begin{cases} \left(\dfrac{\hat{p}_a(\hat{\tau}_{MLE}) - p_0}{p_a^{as} - p_0} \right)^{\left(\frac{\hat{p}_a(\hat{\tau}_{MLE})}{p_0} \right)}, & \text{if } \hat{p}_a(\hat{\tau}_{MLE}) - p_0 \le p_a^{as} - p_0, \\ \left(\dfrac{p_a^{as} - p_0}{\hat{p}_a(\hat{\tau}_{MLE}) - p_0} \right)^{\left(\frac{\hat{p}_a(\hat{\tau}_{MLE})}{p_0} \right)}, & \text{if } \hat{p}_a(\hat{\tau}_{MLE}) - p_0 > p_a^{as} - p_0. \end{cases}$$

Proposed method

Hence, we obtain the proposed estimator of the change point as

$$\hat{\tau}_{NEW}(\hat{w}) = \hat{w}\hat{\tau}_{CUSUM} + (1-\hat{w})\hat{\tau}_{MLE}, \quad 0 \leq \hat{w} \leq 1.$$

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Simulations

$$X_1, X_2, \ldots, X_\tau \overset{i.i.d}{\sim} Bin(n, p_0), \quad X_{\tau+1}, X_{\tau+2}, \ldots, X_T \overset{i.i.d}{\sim} Bin(n, p_a^{True})$$

- We assume the true change point $\tau = 100$.
- The in-control fraction nonconforming $p_0 = 0.1$.
- The out-of-control fraction nonconforming

$p_a^{True} \in \{0.11, 0.12, \cdots, 0.25, 0.30\}$.

Simulations

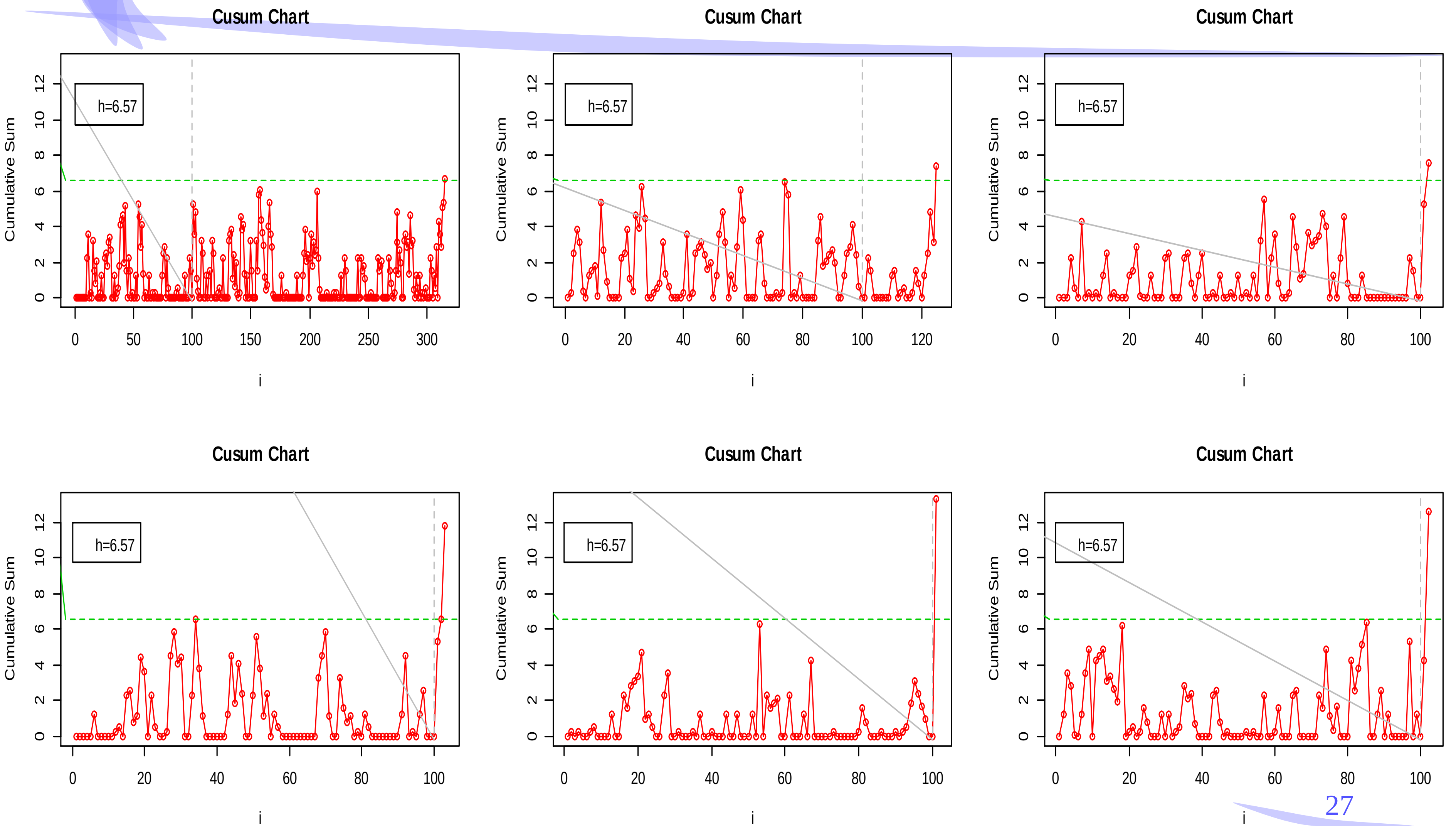
- Consider CUSUM charts designed to detect 30% increase in by setting $p_a^{as} = 1.3 \times p_0 = 0.13$.

- Choose $h = 6.57$ or $h = 11.42$ such that in-control process average run lengths (ARL) is close to 150 or 370.

Simulations

$$p_0 = 0.1, \quad p_a^{as} = 0.13,$$

$$p_a^{True} = 0.1, 0.11, 0.15, 0.20, 0.25, 0.30$$



Simulations

$$p_0 = 0.1,\quad p_a^{as} = 0.13,$$

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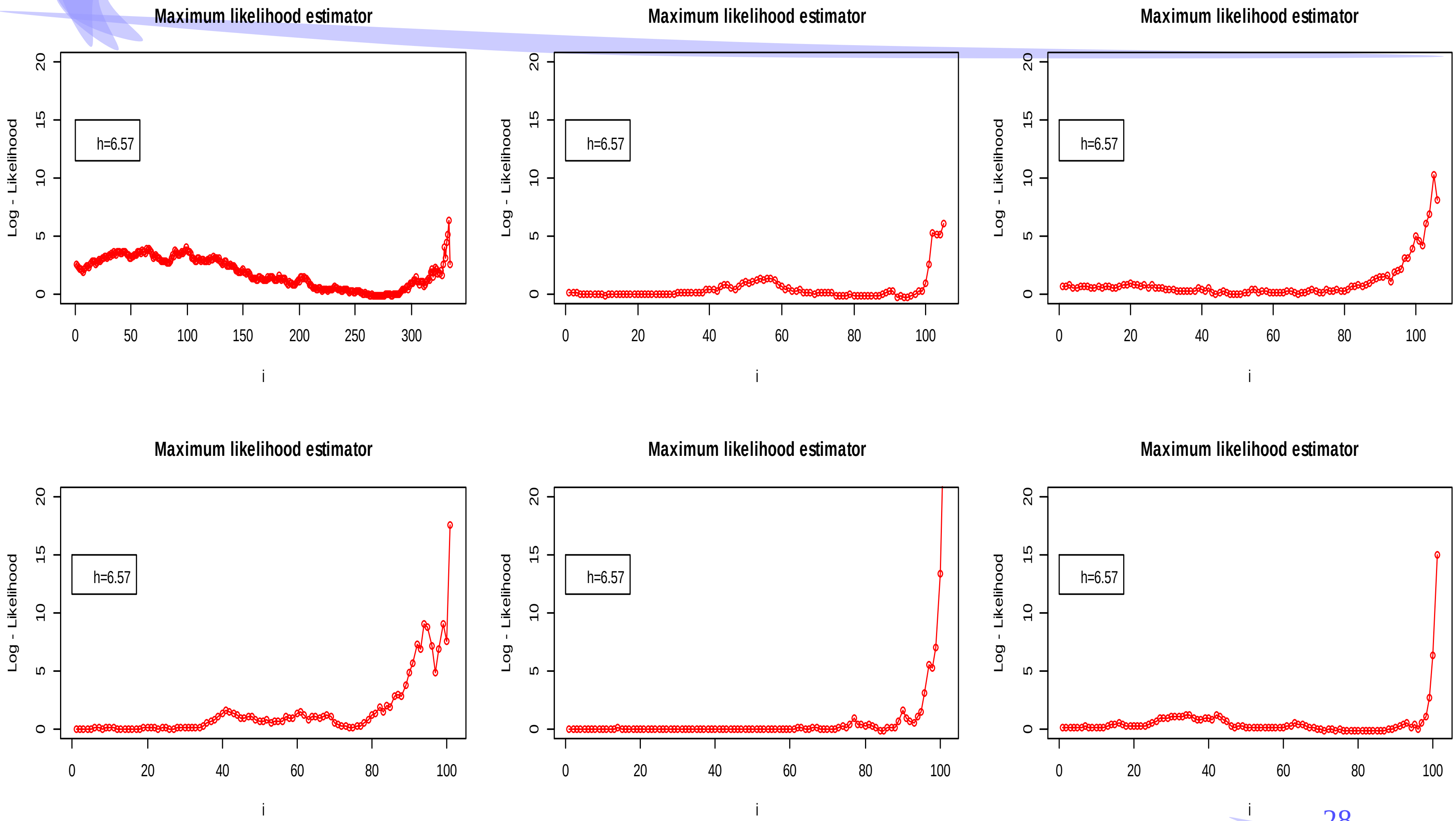

Simulations

$$p_0 = 0.1, \quad p_a^{as} = 0.13,$$

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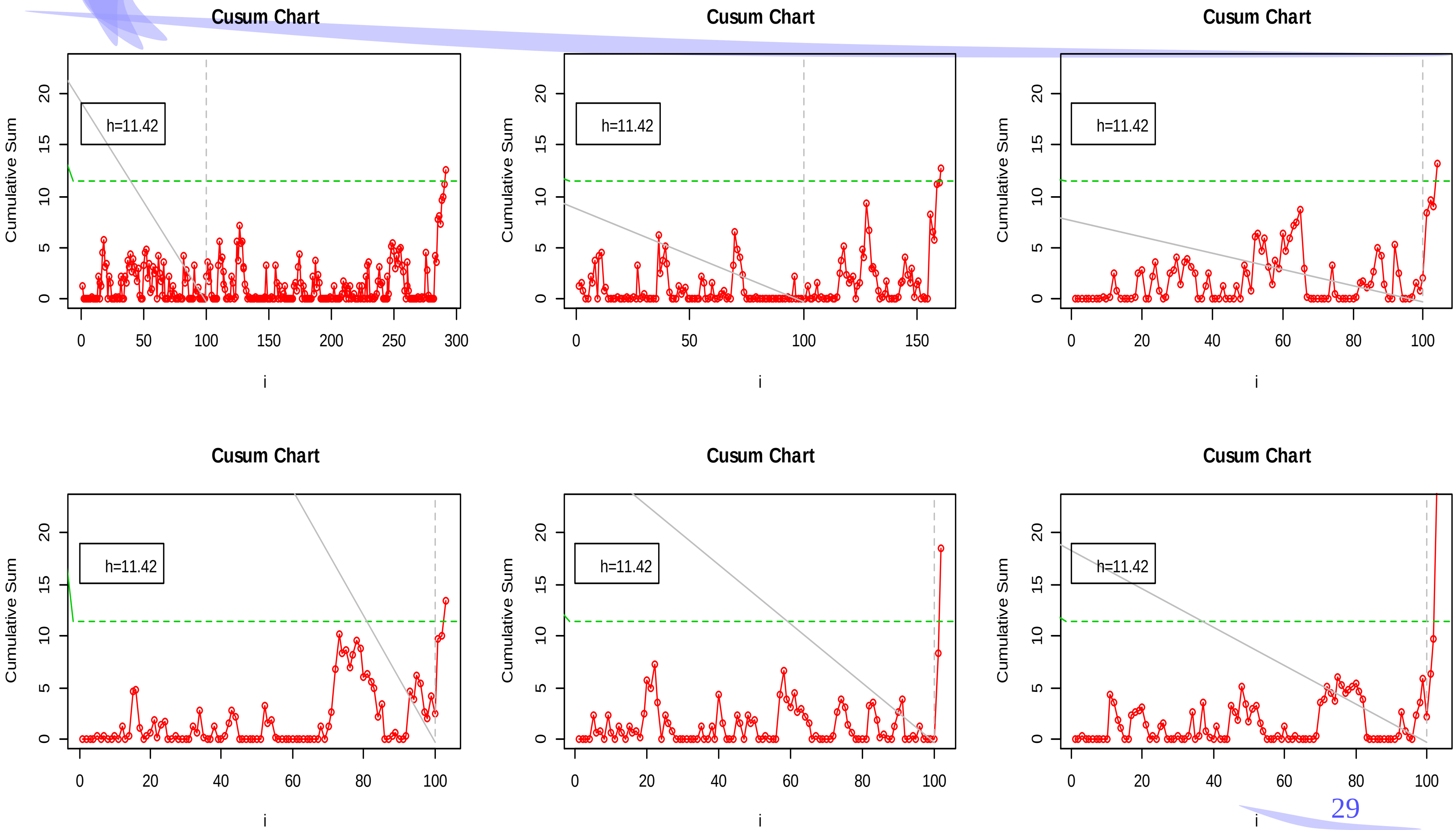
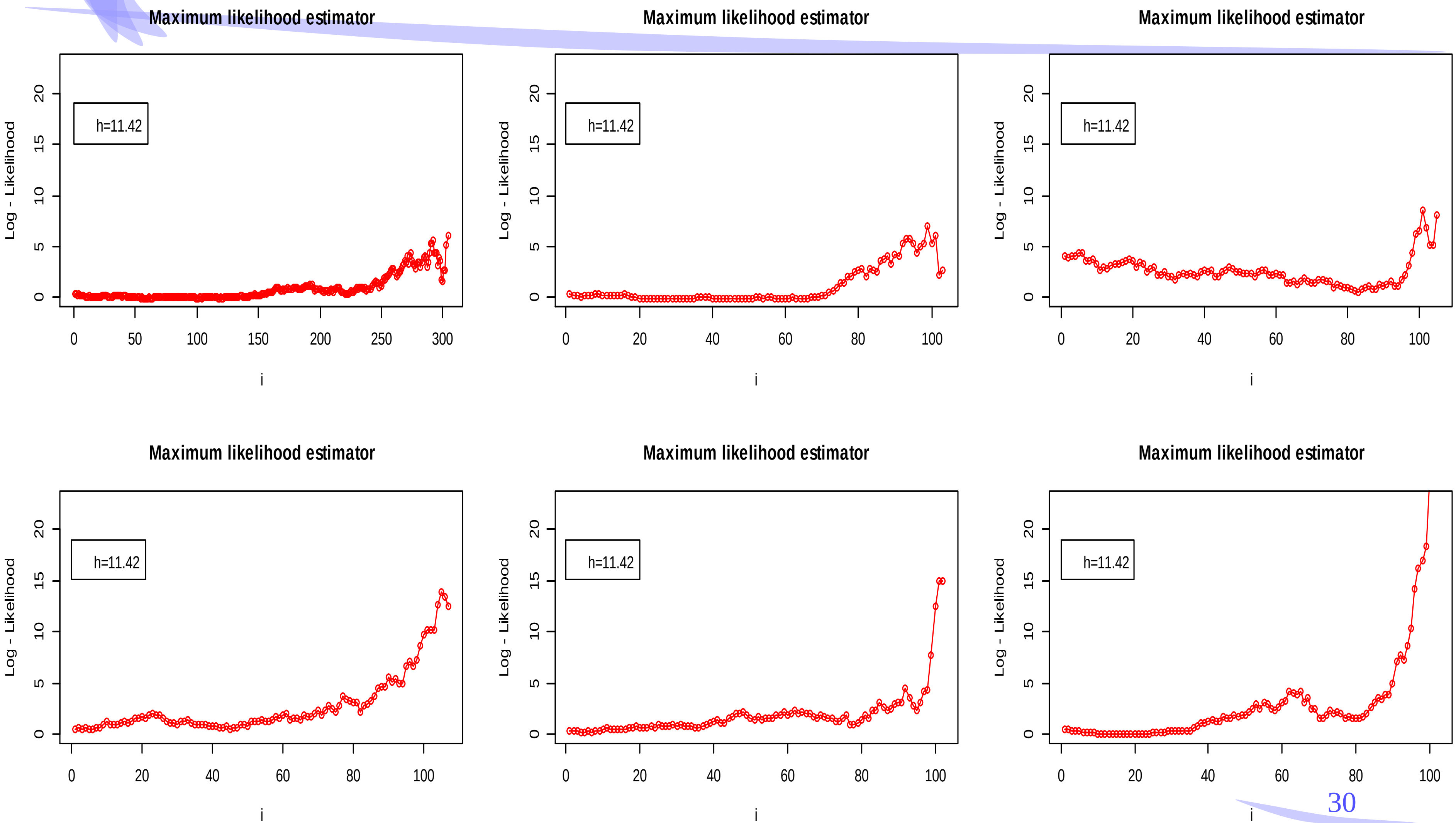


Simulations

$$p_0 = 0.1, \quad p_a^{as} = 0.13,$$

$$p_a^{True} = 0.1, 0.11, 0.15, 0.20, 0.25, 0.30$$



Simulations

Table 4 Simulation results for estimating the change point τ under $n = 50$, $p_0 = 0.1$ and $p_a^{as} = 0.13$ based on 1000 simulation runs. The true change point $\tau = 100$.

$h = 6.57$	p_a^{True}	$E(\hat{\tau}_{MLE})$	$E(\hat{\tau}_{CUSUM})$	$E(\hat{\tau}_{NEW}(w))$	$E(\hat{\tau}_{NEW}(\hat{w}))$
	0.11	118.90	115.98	118.03	117.61
	0.12	108.50	105.51	106.66	107.32
	$0.13 (= p_a^{as})$	104.10	101.33	101.33	103.08
	0.14	102.51	99.94	100.84	101.82
	0.15	101.54	99.11	100.41	100.87
	0.16	101.13	98.78	100.35	100.52
	0.17	100.63	98.47	100.11	100.17
	0.18	100.55	98.54	100.21	100.15
	0.19	100.16	98.45	99.95	99.88
	0.20	100.26	98.42	100.09	99.96
	0.25	100.11	98.34	100.08	99.88
	0.30	100.07	98.44	100.07	99.91

Simulations

$h = 11.42$	p_a^{True}	$E(\hat{\tau}_{MLE})$	$E(\hat{\tau}_{CUSUM})$	$E(\hat{\tau}_{NEW}(w))$	$E(\hat{\tau}_{NEW}(\hat{w}))$
	0.11	153.07	147.96	151.54	150.93
	0.12	115.53	108.84	111.42	113.33
	$0.13\,(= p_a^{as})$	107.14	100.97	100.97	105.20
	0.14	103.79	98.63	100.34	102.25
	0.15	102.63	97.93	100.45	101.38
	0.16	101.77	97.75	100.44	100.81
	0.17	101.50	97.23	100.49	100.64
	0.18	101.13	97.26	100.46	100.44
	0.19	100.88	97.19	100.42	100.22
	0.20	100.62	97.02	100.30	100.11
	0.25	100.30	97.24	100.25	100.03
	0.30	100.14	97.30	100.13	99.98

Simulations

Table 5 Simulation results for estimating the mean squared error the change point τ Under $n=50$, $p_0=0.1$ and $p_a^{as}=0.13$ based on 1000 simulation runs. The true change point $\tau=100$.

$h=6.57$	p_a^{True}	$MSE(\hat{\tau}_{MLE})$	$MSE(\hat{\tau}_{CUSUM})$	$MSE(\hat{\tau}_{NEW}(w))$	$MSE(\hat{\tau}_{NEW}(\hat{w}))$
	0.11	814.11	669.15	759.25	735.72
	0.12	214.27	127.46	145.85	158.91
	0.13 ($=p_a^{as}$)	96.686	30.862	30.862	50.698
	0.14	56.855	16.726	16.838	22.322
	0.15	43.172	10.166	15.404	16.536
	0.16	26.780	9.2780	13.807	8.5762
	0.17	17.391	11.063	10.945	6.4579
	0.18	10.060	9.7160	7.2139	3.8668
	0.19	9.8322	9.3510	18.894	7.1571
	0.20	9.6440	9.4110	8.0442	2.8040
	0.25	2.2600	8.2400	2.1866	1.0691
	0.30	0.2660	8.1890	0.2642	0.3753

Simulations

$h = 11.42$	p_s^{True}	$MSE(\hat{\tau}_{MLE})$	$MSE(\hat{\tau}_{CUSUM})$	$MSE(\hat{\tau}_{NEW}(w))$	$MSE(\hat{\tau}_{NEW}(\hat{w}))$
	0.11	6119.14	5743.80	5835.6	5758.0
	0.12	639.60	374.17	423.99	495.05
	0.13 $(= p_a^{as})$	226.71	60.876	60.876	122.26
	0.14	108.40	36.832	32.666	49.398
	0.15	53.433	31.370	24.371	28.747
	0.16	20.426	24.352	11.983	13.635
	0.17	12.091	30.333	7.8838	8.5666
	0.18	12.338	27.753	8.9755	6.8706
	0.19	4.5920	30.488	3.6264	4.4526
	0.20	5.6930	29.692	4.9255	4.3549
	0.25	0.9280	26.806	0.8860	1.5221
	0.30	0.4600	27.475	0.4562	0.6258

Simulations

- In most cases, the $\hat{\tau}_{NEW}(\hat{w})$ outperforms other estimators.

- Despite some of the mean squared error of $\hat{\tau}_{NEW}(\hat{w})$ in all cases are not the smallest, $\hat{\tau}_{NEW}(\hat{w})$ provides very precise estimator of change point.

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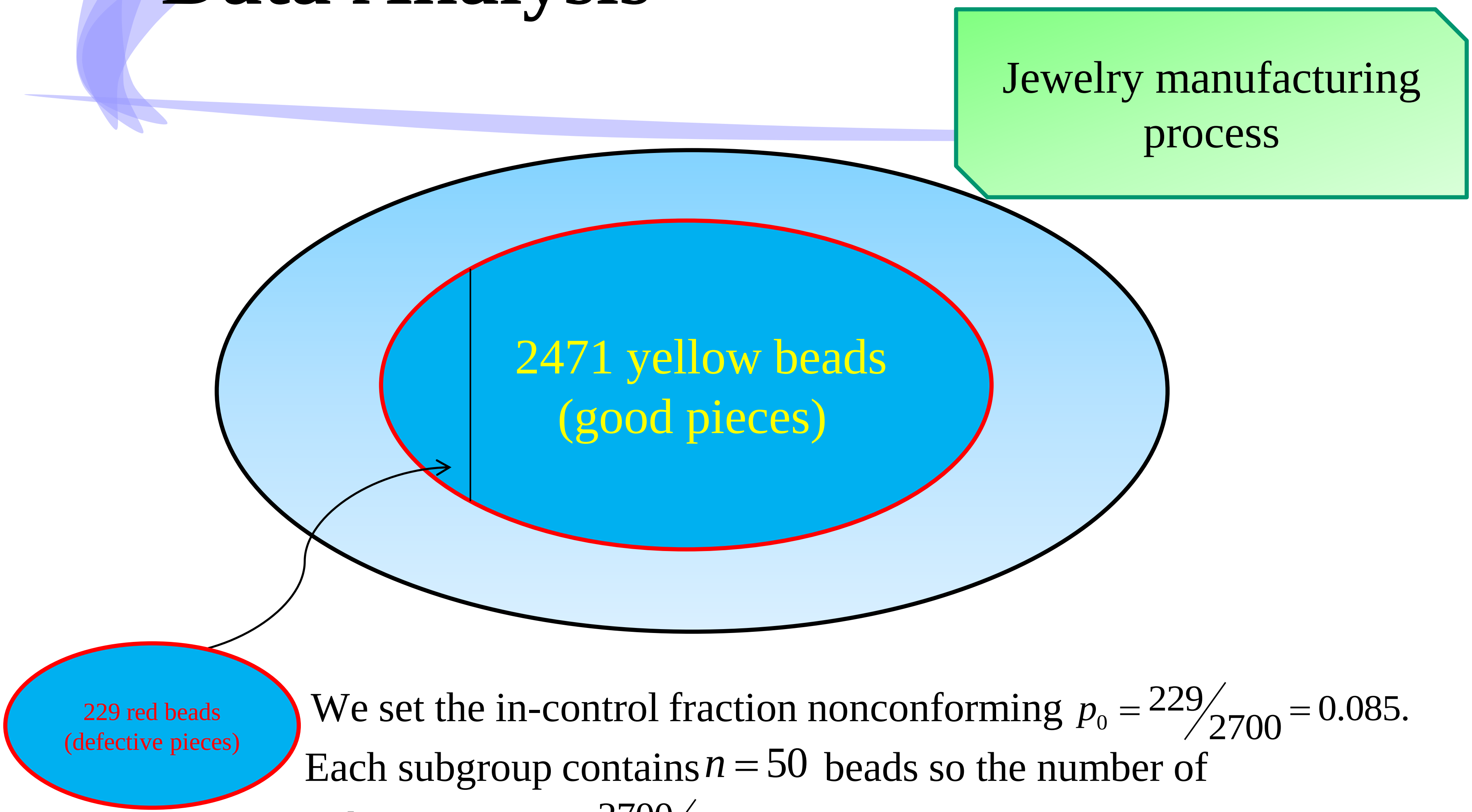
Data Analysis



We set the in-control fraction nonconforming $p_0 = {}^{229}\!/_{2700} = 0.085.$

Each subgroup contains $n = 50$ beads so the number of subgroups is $T = {}^{2700}\!/_{50} = 54.$

$X_i =$ the number of defective pieces in 50 beads, for $i = 1, 2, \ldots, 54.$

Sample subgroup	Defectives x_i	Fraction defective	S_i	Sample subgroup	Defectives x_i	Fraction defective	S_i
1	1	0.02	0	28	2	0.04	0
2	3	0.06.	0	29	4	0.08	0
3	2	0.04	0	30	4	0.08	0
4	3	0.06	0	31	5	0.10	0.1489
5	3	0.06	0	32	5	0.10	0.2978
6	3	0.06	0	33	5	0.10	0.4468
7	2	0.04	0	34	4	0.08	0
8	3	0.06	0	35	3	0.06	0
9	3	0.06	0	36	7	0.14	2.1489
10	4	0.08	0	37	7	0.14	4.2978
11	3	0.06	0	38	3	0.06	2.4468
12	5	0.10	0.1489	39	3	0.06	0.5957
13	3	0.06	0.	40	4	0.08	0
14	4	0.08	0	41	5	0.10	0.1489
15	4	0.08	0	42	7	0.14	2.2978
16.	2	0.04	0	43	2	0.04	0
17	3	0.06	0	44	6	0.12	1.1489
18	6	0.12	1.1489	45	5	0.10	1.2978
19	3	0.06	0	46	7	0.14	3.4468
20	7	0.14	2.1489	47	4	0.08	2.5957
21.	2	0.04	0	48	5	0.10	2.7447
22	3	0.06	0	49	6	0.12	3.8936
23	3	0.06	0	50	7	0.14	6.0426
24	3.	0.06	0	51	8	0.16	9.1915
25	3	0.06	0.	52	6	0.12	10.340
26	3	0.06	0	53	8	0.16	13.489
27	4	0.08.	0	54	9	0.18	17.638

Data Analysis

- Center line $= np = 50 \cdot 0.085 = 4.25.$

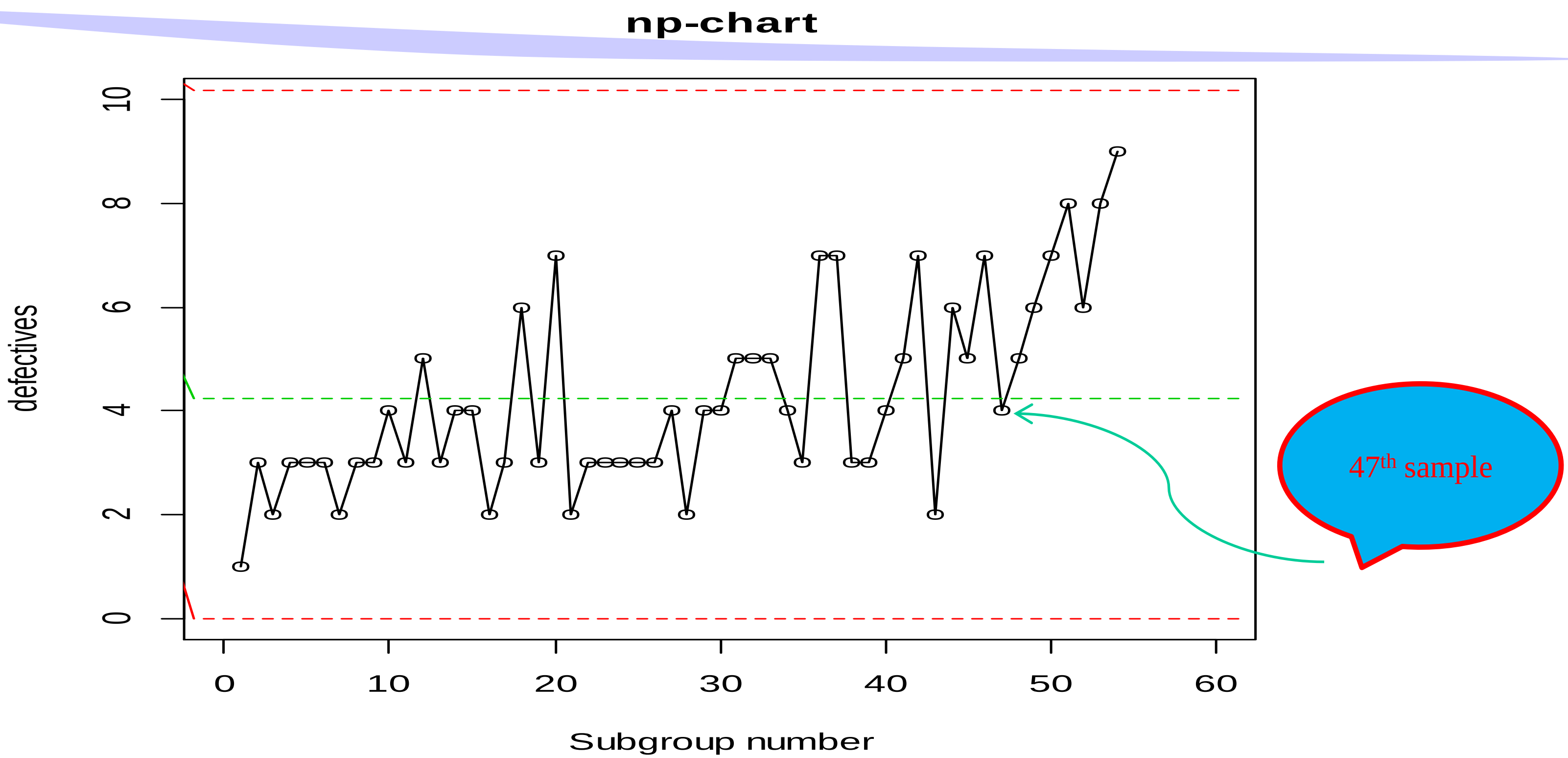
- Upper control limit $= np + 3\sqrt{n \cdot p_0 \cdot (1 - p_0)}$

  $= 4.25 + 3\sqrt{50 \cdot 0.085 \cdot 0.915} = 10.165.$

- Lower control limit $= np - 3\sqrt{n \cdot p_0 \cdot (1 - p_0)}$

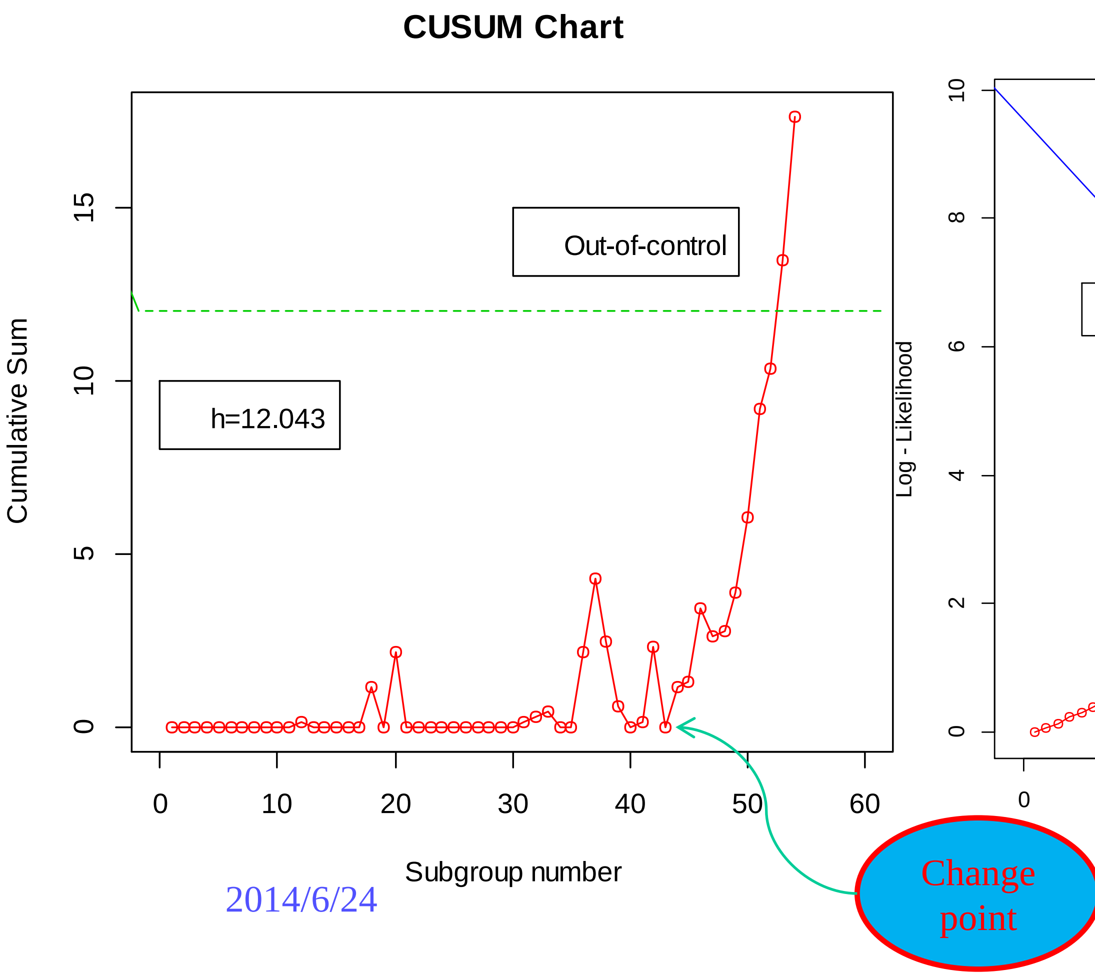
  $= 4.25 - 3\sqrt{50 \cdot 0.085 \cdot 0.915} = -1.665.$

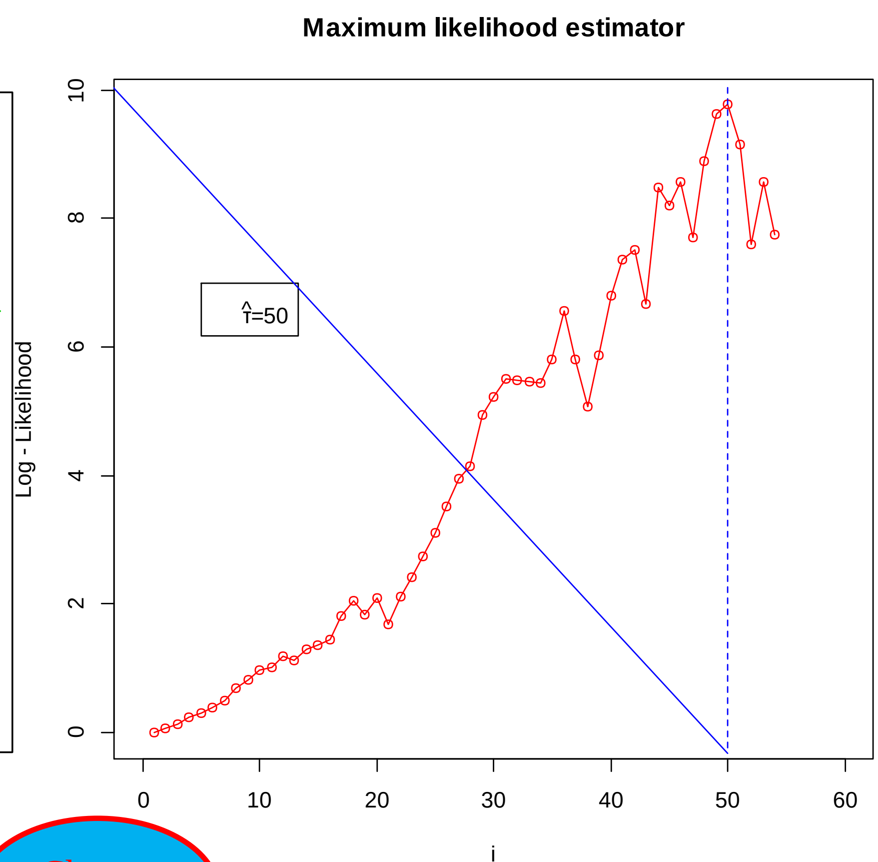
Data Analysis



Although *np*-chart do not detect the process out-of-control, the data slightly increase after the subgroup 40.

Data Analysis





Data Analysis

- We obtain $\hat{\tau}_{CUSUM} = 43$, $\hat{\tau}_{MLE} = 50$ and $\hat{\tau}_{NEW}(\hat{w}) = 47.70983$.

- $\hat{\tau}_{CUSUM}$ always underestimate the true change point while $\hat{\tau}_{MLE}$ slightly overestimate τ when $p_a^{True} > P_a^{as}$.

- Our proposed method may provide more unbiased estimator.

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Conclusion

- The estimator $\hat{\tau}_{NEW}(\hat{w})$ contains advantage of $\hat{\tau}_{CUSUM}$ and $\hat{\tau}_{MLE}$.
- Our proposed method are unbiasedness for estimating the true change point.
- The estimator $\hat{\tau}_{NEW}(\hat{w})$ is more robust than $\hat{\tau}_{CUSUM}$ and $\hat{\tau}_{MLE}$ under different parameter setting.

References

- Assareh H, Mengersen K, Change point Estimation in Monitoring Survival Time. PloS ONE 2012; **7**(3): e33630. Doi:10.1371/journal.pone.0033630.
- Assareh H, Mengersen K, Change point detection in risk adjusted control charts. *Statistical Methods in Medical Reasearch* 2011; 1-22, doi: 10.1177/0962280211426356.
- Burr WI, *Elementary Statistical Quality Control* (1st edn) Milwaukee: New York and Basel, 1979.
- Duran RI, Albin SL. Monitoring a fraction with easy and reliable settings of the false alarm rate. *Quality and Reliability Engineering International* 2009; **25**(8): 1026-1043.
- Emura T, Lin YS, A comparison of normal approximation rules for attribute control charts. *Quality and Reliability Enginerring International* 2013, doi: 10.1002/qre.1601.
- Fuh CD, Mei Y (2008) Optimal stationary binary quantizer for decentralized quickest change detecction in hidden Markov models, 11th International Conference on IEEE Information Fusion.
- Hawkins DM, Olwell DH, Cumulative sum charts and charting for quality improvement (1st edn). Wiley: New York, 1998.
- Khan RA, A note on estimating the mean of a normal distribution with known coefficient of variation. *Journal of the American Statistical Association* 1968; 1039-1041, dol:10.2307/2283896.

References

- Montgomery DC. *Introduction to Statistical Quality Control*. Wiley: New York, NY, 2009.
- Page ES, Continuous inspection schemes. *Biometrika* 1954; **41**:100-114.
- Page ES, A test for a change in a parameter occurring at an unknown point. *Biometrika* 1955; 523-527.
- Pignatiello JJ Jr, Samuel TR, Identifying the time of a step change in the process fraction nonconforming. *Quality Engineering* 2001; **13**(3): 357-365.
- Pignatiello JJ Jr, Perry MB, Estimation of the change point of the process fraction nonconforming in SPC applications. *International Journal of Reliability Quality and Safety Engineering* 2005; **12**: 95-110.
- Pignatiello JJ Jr, Simpson JR, Perry MB, Estimating the change point of the process fraction nonconforming with a monotonic change disturbance in SPC. *Quality and Reliability Engineering Inernational* 2007; **23**: 327-339.
- Rossi G, Sarto SD, Marchi M, A new risk-adjusted Bernoulli cumulative sum chart for monitoring binary health data. *Statistical Methods in Medical Reasearch* 2014; 1-10, dol: 10.1177/0962280214530883.

References

- Wetherill GB, Brown DB, *Statistical Process Control.* Theory and Practice, 1991.
- Wang H, Comparison of p control charts for low defective rate. *Computational Statistics & Data Analysis* 2009; **53**:4210-4220.
- Wang YH, Economic design of CUSUM chart with variable sampling. Master thesis, NTHU library, 2008.
- Wald A, *Sequential Analysis* (1st edn). Wiley: New York, 1947.
- Yang SF, Cheng TC, Hung YC, Cheng SW. A new chart for monitoring service process mean. *Quality and Reliability Engineering International* 2012; **28**: 377-386.

Thanks for your attention .