Dependence measures and competing risks models under the generalized Farlie-Gumbel-Morgenstern copula

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Outline

- Introduction
- Background
- Dependence measures under the generalized FGM copula
- Competing risks analysis under the generalized FGM copula model
- Simulation
- Data analysis
- Concluding remarks

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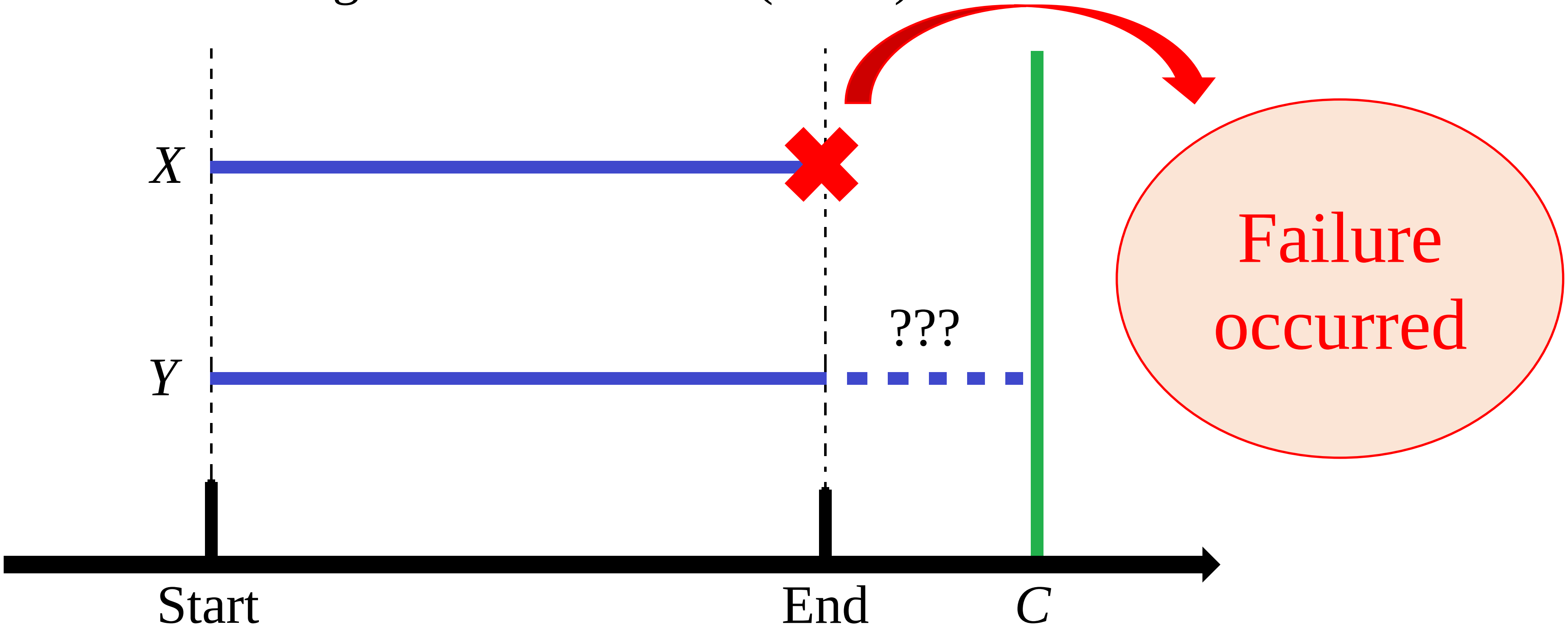
Radio transmitter-receivers data (Mendenhall and Hader 1958)

X : Confirmed failure time (hours)

Y : Unconfirmed failure time (hours)

C : Censoring time fixed at 630 (hours)



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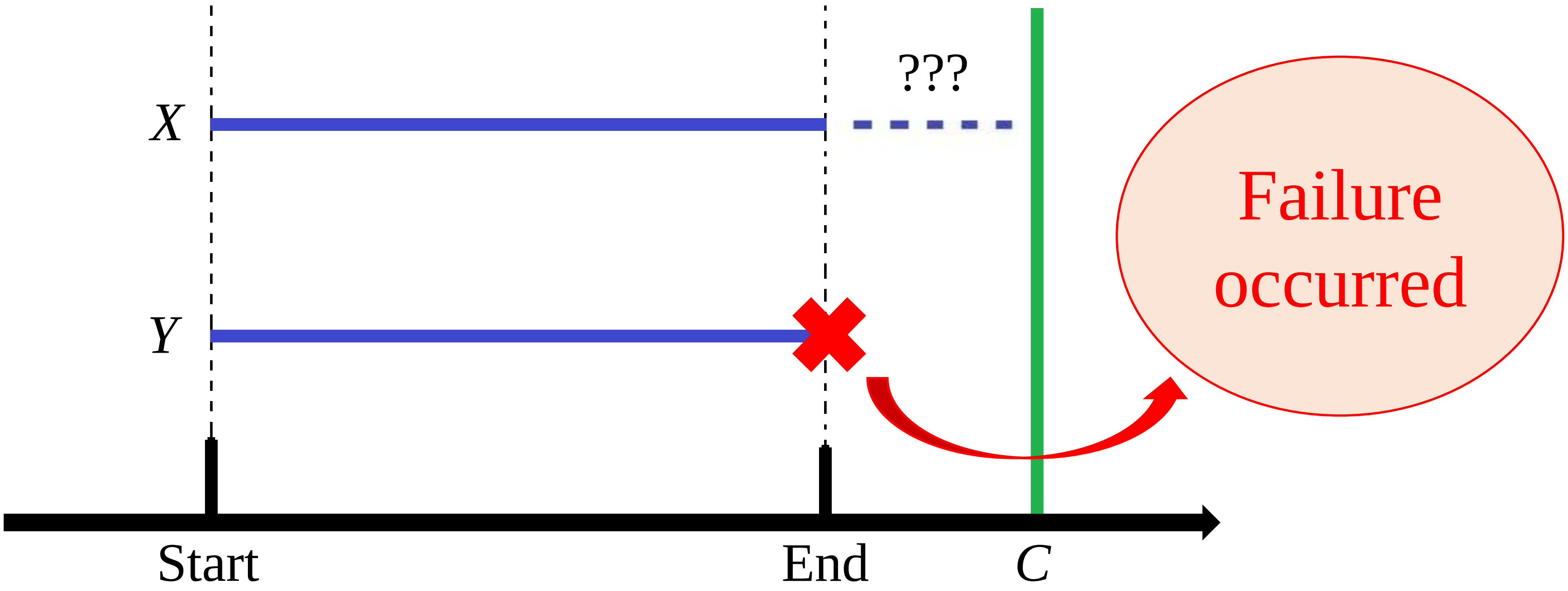

Radio transmitter-receivers data (Mendenhall and Hader 1958)

- **Different types of failure**

→ Competing risks

- **Independent? Dependent?**

→ Copula

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Copula

- A bivariate copula is a bivariate distribution function

$$C_\theta : [0,1]^2 \mapsto [0,1],$$

where θ is a parameter.

- For a bivariate distribution F with marginal F_1 and F_2 ,

one has a unique representation

$$F(x, y) = C_\theta(F_1(x), F_2(y)).$$

Sklar's Theorem (Sklar 1959).

Spearman's rho and Kendall's tau

- Dependence measures based on the concept of concordance.
- Spearman's rho

$$\rho = 12\int_0^1\int_0^1 C(u,v)\,du\,dv - 3.$$

- Kendall's tau

$$\tau = 4\int_0^1\int_0^1 C(u,v)\,dC(u,v) - 1.$$

- Only depend on copula function!

Farlie-Gumbel-Morgenstern Copula

- The Farlie-Gumbel-Morgenstern (FGM) distribution is first introduced by Morgenstern (1956).

- The one parameter FGM copula function is

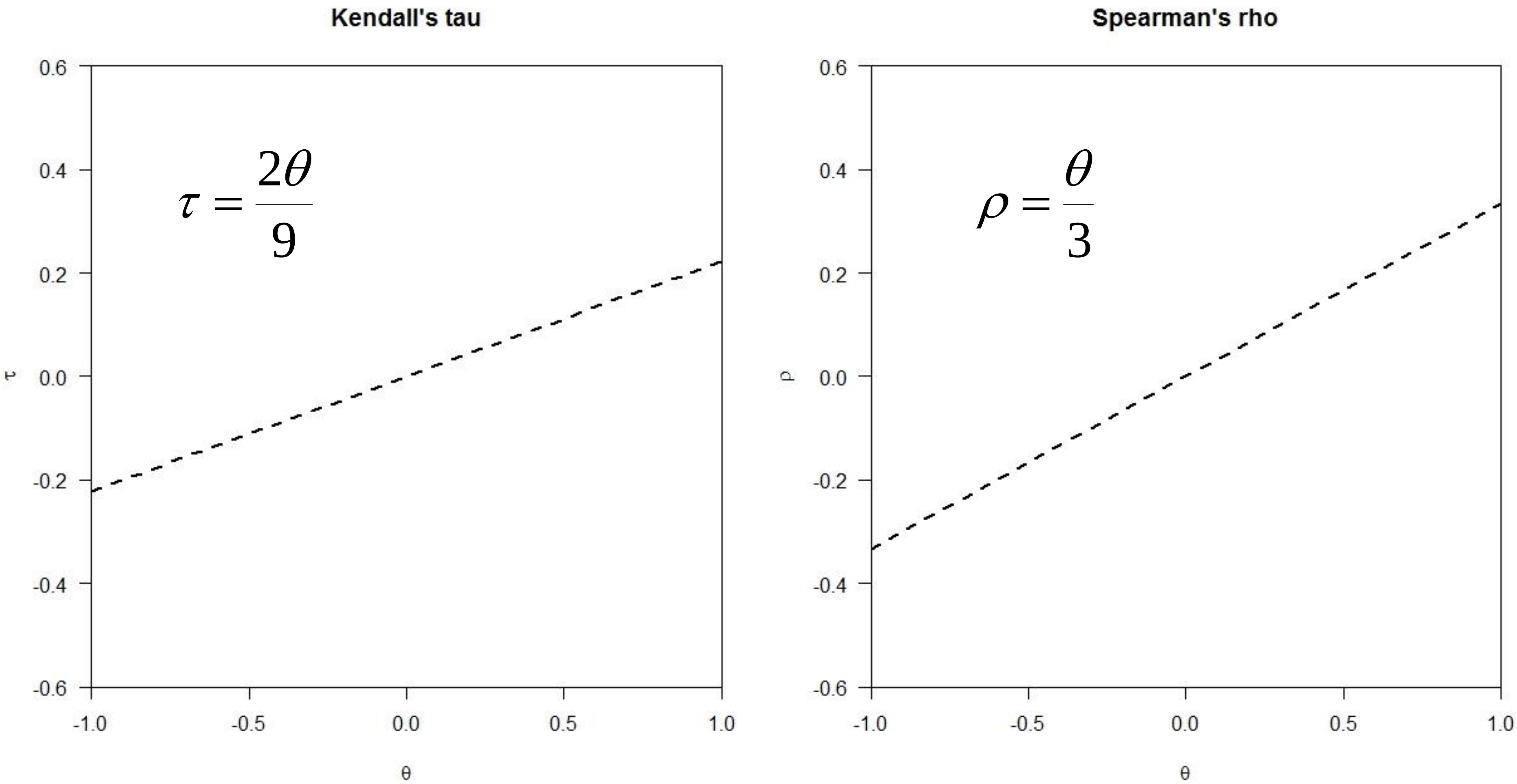
$$C(u,v) = uv\{1+\theta(1-u)(1-v)\},$$

where $\theta \in [-1,1]$. The copula density is

$$c(u,v) = 1+\theta(1-2u)(1-2v).$$

Dependence parameter

Farlie-Gumbel-Morgenstern Copula



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Generalized FGM copula

- The generalized FGM copula (Bairamov and Kotz 2002) is

$$C(u,v) = uv\{1+\theta(1-u^p)^q(1-v^p)^q\}.$$

Dependence parameter

- The copula density is

$$c(u,v) = 1+\theta(1-u^p)^{q-1}\{1-(1+pq)u^p\}^{q-1} \times (1-v^p)^{q-1}\{1-(1+pq)v^p\}^{q-1},$$

where

$$p,q \geq 1 \text{ and } -\min\left\{1, \frac{1}{p^{2q}}\left(\frac{1+pq}{q-1}\right)^{2q-2}\right\} \leq \theta \leq \frac{1}{p^q}\left(\frac{1+pq}{q-1}\right)^{q-1}.$$

Spearman's rho and Kendall's tau

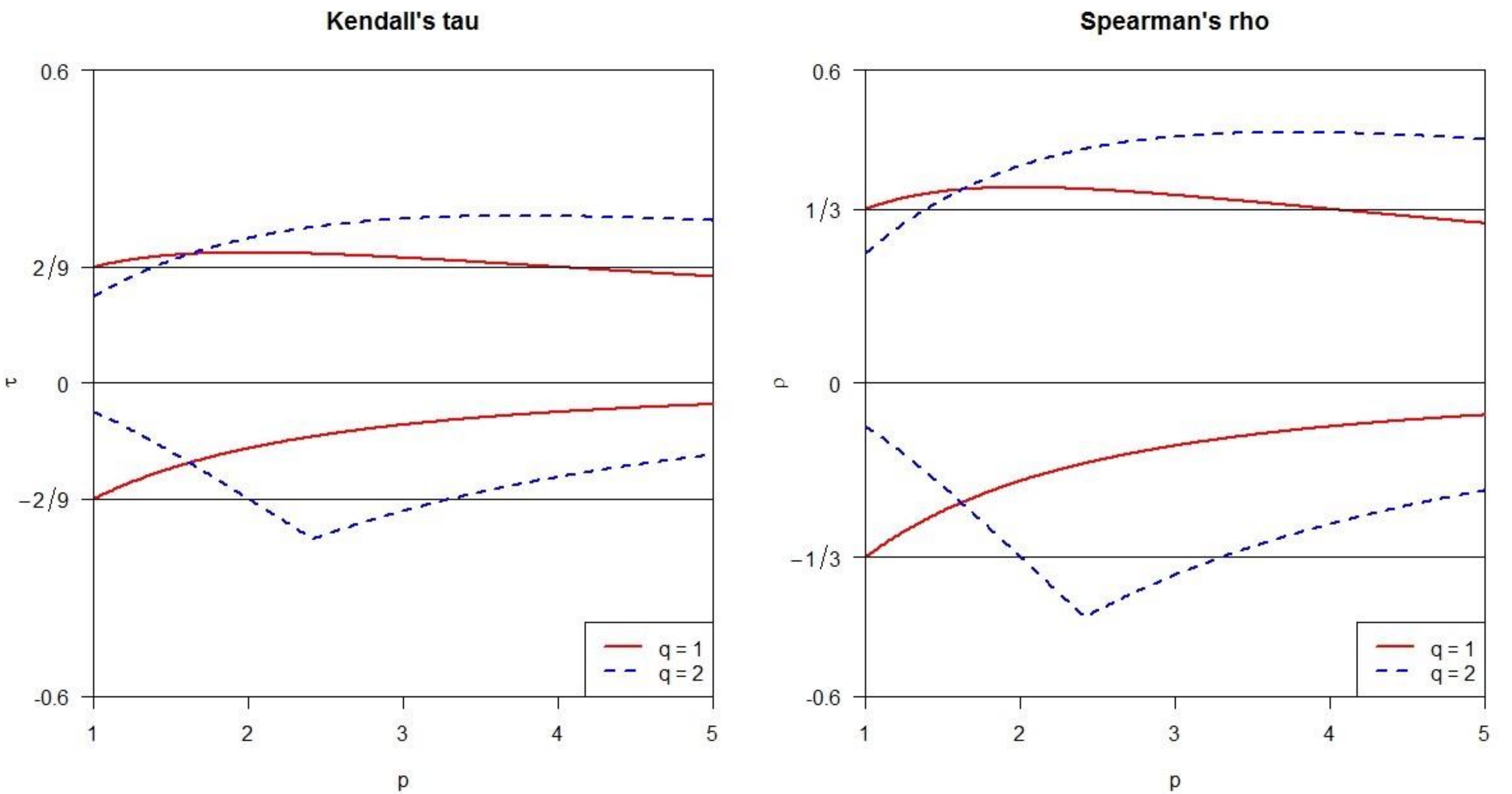
Proposition 1 *Under the generalized FGM copula, Spearman's rho and Kendall's tau are*

$$\rho = 12\left\{\frac{q}{2+pq}B\left(\frac{2}{p},q\right)\right\}^2\theta, \quad \tau = 8\left\{\frac{q}{2+pq}B\left(\frac{2}{p},q\right)\right\}^2\theta.$$

Hence, we have $2\rho = 3\tau$ *in the range of* θ.

- Simplification from the results of Amini et al. (2011)
- Consequence from the more general results of Domma and Giordano (2013, 2016).

Spearman's rho and Kendall's tau



Other dependence measures

- Blest's coefficient and symmetrized version **(Theorem 1)**.

  (rank correlation emphasizes difference in the top ranks)

- Relationship between Blest's coefficient and Spearman's rho **(Corollary 1)**.

- Kochar and Gupta's dependence measures **(Theorem 2)**.

  (base on the concept of quadrant dependence)

- Simplification from the results of Amini et al. (2011).

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Copula model

- Let X and Y be continuous failure times following the generalized FGM copula model

$$F(x, y) = F_1(x)F_2(y)\{1+\theta(1-F_1(x)^p)^q(1-F_2(y)^p)^q\}.$$

- What we actually observed are the first occurring failure time

$$T = \min(X, Y)$$

and the cause of failure

$$\Delta = 1 \text{ if } X \le Y \quad \text{or} \quad \Delta = 2 \text{ if } Y < X.$$

Bivariate distribution

- Traditionally, specific bivariate distribution have been employed to model the joint distribution of X and Y by various authors
    - Bivariate normal (Basu and Ghosh 1978)
    - Bivariate log-normal distribution (Fan and Hsu 2015)
    - Bivariate Weibull distribution (Moeschberger 1974)
- We follow a copula-based approach (Zheng and Klein 1995).

Sub-distribution (density)

- What is sub-distribution (density)?

$Y < X$

$T = \min(X, Y)$

$$F(2,t) = \Pr(\Delta = 2, T \le t) = \int_0^t f(2,z)dz$$

$$f(2,t) = -\frac{\partial \overline{F}(x,y)}{\partial y}\bigg|_{x=y=t}$$

Risk of failure at time t of cause 2

Sub-distribution (density)

Theorem 3 *Under the generalized FGM copula, we derive the general form of sub-densities.*

Theorem 4 *Under the generalized FGM copula, we derive the general form of sub-distribution functions.*

- We examine the cases where sub-distribution functions have closed form.

Burr III distribution (Burr 1942)

- Widely used by various authors
  - Diameter distribution of forest stand
    (Lindsay et al. 1996)

  - Lifetime data analysis with competing risks
    (Crowder 2001; Escarela and Carrière 2003; Lawless 2003)

  - Reliability measure
    (Domma and Giordano 2013)

Burr III distribution (Burr 1942)

- The Burr III marginal distributions defined as

$$F_1(x) = (1 + x^{-\gamma})^{-\alpha}, \quad F_2(y) = (1 + y^{-\gamma})^{-\beta},$$

where (α, β, γ) are all positive parameters.

- Under the generalized FGM copula model with Burr III margins, the bivariate distribution function is

$$F(x, y) = (1 + x^{-\gamma})^{-\alpha}(1 + y^{-\gamma})^{-\beta}$$
$$\times [1 + \theta\{1 - (1 + x^{-\gamma})^{-\alpha p}\}^{q}\{1 - (1 + y^{-\gamma})^{-\beta p}\}^{q}].$$

Burr III distribution (Burr 1942)

- The Burr III marginal distributions defined as

$$F_1(x) = (1 + x^{-\gamma})^{-\alpha}, \quad F_2(y) = (1 + y^{-\gamma})^{-\beta},$$

where (α, β, γ) are al

Closed form exist when
shape parameter is the same !

- Under the
the bivaria

$$F(x, y) = (1 + x$$

$$\times [1 + \theta \{ 1 - (1 + x \quad \} \{ 1 - (1 + y^{-\gamma})^{-\beta p} \}^{q}].$$

Sub-density

Theorem 5 *Under the generalized FGM copula model with Burr III margins, the sub-density is*

$$f(2,t)=\beta K_\gamma(t)H_\gamma(t)^{\beta+1}-\beta K_\gamma(t)H_\gamma(t)^{\alpha+\beta+1}$$
$$-\theta\beta\sum_{i=0}^{q}\sum_{j=0}^{q-1}\binom{q}{i}\binom{q-1}{j}(-1)^{i+j}\{K_\gamma(t)H_\gamma(t)^{\alpha(pi+1)+\beta(pj+1)+1}$$
$$-(1+pq)K_\gamma(t)H_\gamma(t)^{\alpha(pi+1)+\beta(pj+p+1)+1}\},$$

where $K_\gamma(t)=\gamma t^{-\gamma-1}$ *and* $H_\gamma(t)=(1+t^{-\gamma})^{-1}$.

Sub-distribution

Theorem 6 *Under the generalized FGM copula model with Burr III margins, the sub-distribution is*

$$F(2,t) = H_\gamma(t)^\beta - \frac{\beta}{\alpha+\beta} H_\gamma(t)^{\alpha+\beta} - \theta\beta \sum_{i=0}^{q} \sum_{j=0}^{q-1} \binom{q}{i} \binom{q-1}{j} (-1)^{i+j}$$

$$\times \left\{ \frac{H_\gamma(t)^{\alpha(pi+1)+\beta(pj+1)}}{\alpha(pi+1)+\beta(pj+1)} - \frac{(1+pq)H_\gamma(t)^{\alpha(pi+1)+\beta(pj+p+1)}}{\alpha(pi+1)+\beta(pj+p+1)} \right\},$$

where $H_\gamma(t) = (1+t^{-\gamma})^{-1}$.

Sub-distribution with Burr III margins

- Let $p = q = 1$ (original FGM) and $t \to \infty$, we have

$$F(2,\infty) = P(Y < X) = \frac{\alpha}{\alpha+\beta} + \theta \frac{\alpha\beta(\alpha-\beta)}{(\alpha+\beta)(2\alpha+\beta)(\alpha+2\beta)}.$$

- Our expressions generalize the reliability measure obtained by Domma and Giordano (2013).

Maximum likelihood inference

- Under the generalize FGM copula model with Burr III margins, the bivariate distribution function of X and Y is

$$F_{\boldsymbol{\varphi}}(x,y)=(1+x^{-\gamma})^{-\alpha}(1+y^{-\gamma})^{-\beta}$$
$$\times[1+\theta\{1-(1+x^{-\gamma})^{-\alpha p}\}^{q}\{1-(1+y^{-\gamma})^{-\beta p}\}^{q}].$$

where $\boldsymbol{\varphi}=(\alpha,\beta,\gamma)$ are all positive parameters with margins

$$F_1(x)=(1+x^{-\gamma})^{-\alpha},\quad F_2(y)=(1+y^{-\gamma})^{-\beta}.$$

Nonidentifiability

- Due to nonidetifiability of competing risks data, we assume

$$\theta, \quad p \in \{1, 2, \cdots\} \quad \text{and} \quad q \in \{1, 2, \cdots\}$$

are given.

Notations

X_i : failure time due to cause 1

Y_i : failure time due to cause 2

C_i : independent censoring time

$T_i = \min(X_i, Y_i, C_i)$: observed failure time

$\delta_i = \mathbf{I}(T_i = X_i)$: indicator of failure cause 1

$\delta_i^* = \mathbf{I}(T_i = Y_i)$: indicator of failure cause 2

Our data is $(T_i, \delta_i, \delta_i^*)$, for $i = 1, 2, \cdots, n$.

Likelihood contribution

Table 1. Three different observation patterns under competing risks.

Observation	δ_i	δ_i^*	Likelihood contribution
Cause 1 failure	1	0	$\Pr(T_i = X_i)$
Cause 2 failure	0	1	$\Pr(T_i = Y_i)$
Censoring	0	0	$\Pr(X_i > T_i, Y_i > T_i)$

Maximum likelihood inference

$$L_n(\varphi) = \prod_{i=1}^{n} f_{\varphi}(1, T_i)^{\delta_i} f_{\varphi}(2, T_i)^{\delta_i^*} \bar{F}_{\varphi}(T_i, T_i)^{1-\delta_i-\delta_i^*}$$

$$\ell_n(\varphi) = \sum_{i=1}^{n} \delta_i \log f_{\varphi}(1, T_i) + \sum_{i=1}^{n} \delta_i^* \log f_{\varphi}(2, T_i)$$

$$+ \sum_{i=1}^{n} (1 - \delta_i - \delta_i^*) \log \bar{F}_{\varphi}(T_i, T_i)$$

$$\hat{\varphi} = \arg\max_{\varphi \in \Theta} \ell_n(\varphi)$$

Randomized Newton-Raphson (Hu and Emura 2015)

Algorithm 1
Randomized Newton-Raphson algorithm

Step1. Set initial value $(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)})$.

Step2. Repeat the Newton-Raphson iterations:

$$\begin{bmatrix} \alpha^{(k+1)} \\ \beta^{(k+1)} \\ \gamma^{(k+1)} \end{bmatrix} = \begin{bmatrix} \alpha^{(k)} \\ \beta^{(k)} \\ \gamma^{(k)} \end{bmatrix} - \begin{bmatrix} \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \alpha^2} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \alpha \partial \gamma} \\ \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \alpha \partial \beta} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \beta^2} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \beta \partial \gamma} \\ \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \alpha \partial \gamma} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \beta \partial \gamma} & \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \gamma^2} \end{bmatrix}^{-1} \begin{bmatrix} \frac{\partial \ell_n(\boldsymbol{\varphi})}{\partial \alpha} \\ \frac{\partial \ell_n(\boldsymbol{\varphi})}{\partial \beta} \\ \frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \gamma} \end{bmatrix} \Bigg|_{\alpha=\alpha^{(k)},\, \beta=\beta^{(k)},\, \gamma=\gamma^{(k)}}$$

- If $\max\{|\alpha^{(k+1)} - \alpha^{(k)}|, |\beta^{(k+1)} - \beta^{(k)}|, |\gamma^{(k+1)} - \gamma^{(k)}|\} < 10^{-5}$, then stop and the MLE is $\hat{\boldsymbol{\varphi}} = (\alpha^{(k+1)}, \beta^{(k+1)}, \gamma^{(k+1)})$.

Randomize conditions

$$\max\{|\alpha^{(k+1)}-\alpha^{(k)}|,|\beta^{(k+1)}-\beta^{(k)}|,|\gamma^{(k+1)}-\gamma^{(k)}|\}>D$$

Diverge

$$f_{\varphi}(1,T_i)=0,\quad f_{\varphi}(2,T_i)=0\quad \text{or}\quad \overline{F}_{\varphi}(T_i,T_i)=0$$

$$\frac{\partial \ell_n(\varphi)}{\partial \alpha}=\sum_{i=1}^{n}\left\{\delta_i\frac{1}{f_{\varphi}(1,T_i)}\frac{\partial f_{\varphi}(1,T_i)}{\partial \alpha}\right\}+\sum_{i=1}^{n}\left\{\delta_i^{*}\frac{1}{f_{\varphi}(2,T_i)}\frac{\partial f_{\varphi}(2,T_i)}{\partial \alpha}\right\}$$

$$+\sum_{i=1}^{n}\left\{(1-\delta_i-\delta_i^{*})\frac{1}{\overline{F}_{\varphi}(T_i,T_i)}\frac{\partial \overline{F}_{\varphi}(T_i,T_i)}{\partial \alpha}\right\},$$

Randomize the initial value

- Return to Step1 with the initial value

$$(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)})$$

replaced by

$$(\alpha^{(0)} \times \exp(u_1), \beta^{(0)} \times \exp(u_2), \gamma^{(0)} \times \exp(u_3)),$$

where

$$u_i \sim U(-r_i, r_i), \quad r_i > 0, \quad i = 1, 2, 3$$

are independent uniform random variables.

Choosing initial values

$$\alpha = E(X)/B\left(\alpha + \frac{1}{\gamma}, 1 - \frac{1}{\gamma}\right)$$

$$\alpha^{(0)} = \bar{X} = \sum_{i=1}^{n} \delta_i T_i / \sum_{i=1}^{n} \delta_i$$

$$\beta = E(Y)/B\left(\beta + \frac{1}{\gamma}, 1 - \frac{1}{\gamma}\right)$$

$$\beta^{(0)} = \bar{Y} = \sum_{i=1}^{n} \delta_i^* T_i / \sum_{i=1}^{n} \delta_i^*$$

Start with $\gamma^{(0)} = 1$, if the algorithm cannot converge, increase the value.

Standard error and confidence interval

- The standard error of $\hat{\alpha}$ is

$$SE(\hat{\alpha}) = \sqrt{\left[\left\{ -\frac{\partial^2 \ell_n(\boldsymbol{\varphi})}{\partial \boldsymbol{\varphi} \partial \boldsymbol{\varphi}^{\mathrm{T}}} \right\}^{-1} \Bigg|_{\boldsymbol{\varphi}=\hat{\boldsymbol{\varphi}}} \right]_{11}}.$$

- The $(1-\varepsilon)\%$ confidence interval of $\hat{\alpha}$ is

$$\hat{\alpha} \times \exp\{ \pm Z_{\varepsilon/2} SE(\log \hat{\alpha}) \},$$

where Z_p is the *p*-th upper quantile of standard normal.

- Similar for $\hat{\beta}$ and $\hat{\gamma}$.

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Simulation design

- Generate samples $\{ (X_i, Y_i), \quad i = 1, 2, \cdots n \}$ from the generalized FGM copula with Burr III margins.

$$(\alpha, \beta, \gamma) = (2, 2, 3), (2, 4, 5) \text{ or } (3, 2, 7)$$

- Consider independent censoring $C_i \sim U(0, w), \quad i = 1, 2, \cdots n.$

- Competing risks data $\{ (T_i, \delta_i, \delta_i^*), \quad i = 1, 2, \cdots n \}$, where

$$T_i = \min(X_i, Y_i, C_i), \quad \delta_i = \mathbf{I}(T_i = X_i), \quad \delta_i^* = \mathbf{I}(T_i = Y_i).$$

- Obtain the MLE by Algorithm 1 with initial value

$$(\alpha^{(0)}, \beta^{(0)}, \gamma^{(0)}) = (\bar{X}, \bar{Y}, 4).$$

Simulation design

- Consider independent censoring $C_i \sim U(0, w), \quad i = 1, 2, \cdots n.$

Table 2. Three censoring percentage with $(\alpha, \beta, \gamma) = (2, 2, 3)$.

		$w=6$	$w=3$	$w=2$
Cause 1 failure	$\Pr(T_i = X_i)$	40%	30%	20%
Cause 2 failure	$\Pr(T_i = Y_i)$	40%	30%	20%
Censoring	$\Pr(T_i = C_i)$	20%	40%	60%
		light	moderate	heavy

Notations

- R: simulation repetitions time.
- $E(\hat{\alpha}) \equiv \frac{1}{R}\sum_{r=1}^{R}\hat{\alpha}_{(r)}$.
- $MSE(\hat{\alpha}) = \frac{1}{R}\sum_{r=1}^{R}(\hat{\alpha}_{(r)} - \hat{\alpha})^2$.
- $SD(\hat{\alpha}) = \sqrt{\frac{1}{R-1}\sum_{r=1}^{R}(\hat{\alpha}_{(r)} - \bar{\hat{\alpha}}_{(\cdot)})^2}$.

Similar for $\hat{\beta}$ and $\hat{\gamma}$.

- AI = average iteration number until convergence.
- AR = average randomization number until converge.
- CP = coverage probability of 95% confidence interval.

Results

Table 3(a). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$E(\hat{\alpha})$	$E(\hat{\beta})$	$E(\hat{\gamma})$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$MSE(\hat{\gamma})$	AI	AR
$\alpha=2$	$T_i=X_i\,(40\%)$	100	2.020	2.032	3.035	0.052	0.053	0.053	4.8	0
$\beta=2$	$T_i=Y_i\,(40\%)$	200	2.014	2.009	3.019	0.025	0.023	0.025	4.8	0
$\gamma=3$	$T_i=C_i\,(20\%)$	300	2.009	2.005	3.015	0.016	0.015	0.015	4.8	0
	$T_i=X_i\,(30\%)$	100	2.022	2.035	3.043	0.064	0.062	0.075	4.8	0
	$T_i=Y_i\,(30\%)$	200	2.015	2.007	3.028	0.031	0.028	0.037	4.8	0
	$T_i=C_i\,(40\%)$	300	2.012	2.006	3.017	0.019	0.017	0.023	4.8	0
	$T_i=X_i\,(20\%)$	100	2.030	2.043	3.060	0.088	0.082	0.136	4.8	0
	$T_i=Y_i\,(20\%)$	200	2.022	2.009	3.030	0.042	0.036	0.067	4.9	0
	$T_i=C_i\,(60\%)$	300	2.011	2.007	3.025	0.024	0.022	0.040	4.9	0

Results

Table 3(b). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$E(\hat{\alpha})$	$E(\hat{\beta})$	$E(\hat{\gamma})$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$MSE(\hat{\gamma})$	AI	AR
$\alpha = 2$	$T_i = X_i$ (59%)	100	2.025	4.122	5.060	0.051	0.352	0.133	5.5	<0.1
$\beta = 4$	$T_i = Y_i$ (21%)	200	2.015	4.050	5.033	0.024	0.141	0.063	5.5	<0.1
$\gamma = 5$	$T_i = C_i$ (20%)	300	2.011	4.030	5.025	0.015	0.089	0.041	5.5	<0.1
	$T_i = X_i$ (45%)	100	2.030	4.144	5.076	0.062	0.463	0.192	5.3	0
	$T_i = Y_i$ (15%)	200	2.017	4.053	5.045	0.029	0.182	0.091	5.2	0
	$T_i = C_i$ (40%)	300	2.014	4.034	5.027	0.018	0.111	0.057	5.1	<0.1
	$T_i = X_i$ (30%)	100	2.037	4.203	5.128	0.086	0.721	0.307	5.1	0
	$T_i = Y_i$ (10%)	200	2.021	4.078	5.057	0.038	0.266	0.152	5.0	0
	$T_i = C_i$ (60%)	300	2.012	4.041	5.038	0.023	0.159	0.090	5.0	0

Results

Table 3(c). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$E(\hat{\alpha})$	$E(\hat{\beta})$	$E(\hat{\gamma})$	$MSE(\hat{\alpha})$	$MSE(\hat{\beta})$	$MSE(\hat{\gamma})$	AI	AR
$\alpha = 3$	$T_i = X_i$ (29%)	100	3.065	2.033	7.076	0.155	0.050	0.253	8.3	1.1
$\beta = 2$	$T_i = Y_i$ (51%)	200	3.021	2.014	7.057	0.065	0.024	0.138	8.6	1.3
$\gamma = 7$	$T_i = C_i$ (20%)	300	3.015	2.006	7.045	0.040	0.015	0.088	8.8	1.5
	$T_i = X_i$ (21%)	100	3.087	2.044	7.117	0.206	0.070	0.385	7.4	0.5
	$T_i = Y_i$ (39%)	200	3.039	2.020	7.073	0.086	0.034	0.194	7.4	0.6
	$T_i = C_i$ (40%)	300	3.033	2.006	7.050	0.058	0.018	0.108	7.2	0.5
	$T_i = X_i$ (14%)	100	3.078	2.044	7.194	0.277	0.087	0.626	6.3	0.1
	$T_i = Y_i$ (26%)	200	3.073	2.012	7.085	0.145	0.039	0.293	6.1	<0.1
	$T_i = C_i$ (60%)	300	3.037	2.013	7.058	0.080	0.025	0.186	6.1	<0.1

Results

Table 4(a). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\alpha})$	$SE(\hat{\alpha})$	$CP(\hat{\alpha})$	$SD(\hat{\beta})$	$SE(\hat{\beta})$	$CP(\hat{\beta})$	$SD(\hat{\gamma})$	$SE(\hat{\gamma})$	$CP(\hat{\gamma})$
$\alpha = 2$	$T_i = X_i$ (40%)	100	0.228	0.218	0.945	0.227	0.220	0.944	0.228	0.224	0.939
$\beta = 2$	$T_i = Y_i$ (40%)	200	0.158	0.154	0.945	0.151	0.153	0.962	0.157	0.158	0.956
$\gamma = 3$	$T_i = C_i$ (20%)	300	0.126	0.125	0.941	0.122	0.125	0.953	0.123	0.129	0.959
	$T_i = X_i$ (30%)	100	0.252	0.239	0.941	0.246	0.241	0.946	0.272	0.269	0.945
	$T_i = Y_i$ (30%)	200	0.177	0.168	0.946	0.167	0.167	0.950	0.191	0.189	0.941
	$T_i = C_i$ (40%)	300	0.137	0.137	0.948	0.131	0.136	0.957	0.151	0.154	0.961
	$T_i = X_i$ (20%)	100	0.296	0.273	0.944	0.284	0.276	0.939	0.364	0.360	0.945
	$T_i = Y_i$ (20%)	200	0.203	0.191	0.941	0.190	0.190	0.954	0.257	0.251	0.943
	$T_i = C_i$ (60%)	300	0.156	0.155	0.953	0.148	0.155	0.965	0.199	0.204	0.952

Results

Table 4(b). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\alpha})$	$SE(\hat{\alpha})$	CP($\hat{\alpha}$)	$SD(\hat{\beta})$	$SE(\hat{\beta})$	CP($\hat{\beta}$)	$SD(\hat{\gamma})$	$SE(\hat{\gamma})$	CP($\hat{\gamma}$)
$\alpha = 2$	$T_i = X_i$ (59%)	100	0.225	0.212	0.950	0.581	0.549	0.949	0.360	0.359	0.947
$\beta = 4$	$T_i = Y_i$ (21%)	200	0.154	0.149	0.947	0.373	0.376	0.958	0.249	0.252	0.953
$\gamma = 5$	$T_i = C_i$ (20%)	300	0.121	0.122	0.956	0.298	0.304	0.957	0.201	0.206	0.952
	$T_i = X_i$ (45%)	100	0.248	0.232	0.939	0.665	0.620	0.950	0.432	0.419	0.935
	$T_i = Y_i$ (15%)	200	0.171	0.163	0.940	0.424	0.419	0.952	0.299	0.293	0.946
	$T_i = C_i$ (40%)	300	0.133	0.133	0.952	0.331	0.338	0.965	0.236	0.239	0.960
	$T_i = X_i$ (30%)	100	0.292	0.263	0.937	0.825	0.744	0.951	0.539	0.540	0.952
	$T_i = Y_i$ (10%)	200	0.193	0.184	0.943	0.510	0.490	0.949	0.386	0.374	0.951
	$T_i = C_i$ (60%)	300	0.151	0.149	0.952	0.397	0.393	0.956	0.298	0.304	0.949

Results

Table 4(c). Simulation results on the MLE of (α, β, γ) based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\alpha})$	$SE(\hat{\alpha})$	CP$(\hat{\alpha})$	$SD(\hat{\beta})$	$SE(\hat{\beta})$	CP$(\hat{\beta})$	$SD(\hat{\gamma})$	$SE(\hat{\gamma})$	CP$(\hat{\gamma})$
$\alpha = 3$	$T_i = X_i$ (29%)	100	0.388	0.369	0.934	0.220	0.217	0.945	0.497	0.505	0.948
$\beta = 2$	$T_i = Y_i$ (51%)	200	0.255	0.255	0.960	0.155	0.152	0.939	0.368	0.356	0.944
$\gamma = 7$	$T_i = C_i$ (20%)	300	0.200	0.207	0.966	0.124	0.123	0.953	0.293	0.289	0.938
	$T_i = X_i$ (21%)	100	0.446	0.418	0.947	0.261	0.243	0.932	0.609	0.591	0.935
	$T_i = Y_i$ (39%)	200	0.290	0.288	0.953	0.184	0.169	0.932	0.434	0.414	0.938
	$T_i = C_i$ (40%)	300	0.239	0.235	0.951	0.134	0.137	0.955	0.324	0.337	0.953
	$T_i = X_i$ (14%)	100	0.521	0.485	0.949	0.291	0.278	0.946	0.767	0.744	0.941
	$T_i = Y_i$ (26%)	200	0.374	0.341	0.935	0.197	0.193	0.947	0.535	0.520	0.949
	$T_i = C_i$ (60%)	300	0.281	0.272	0.945	0.159	0.157	0.949	0.428	0.420	0.941

Results

Table 5(a). Simulation results one the MLE of the mean failure time $\mu_X = E(X)$ and $\mu_Y = E(Y)$ based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\mu}_X)$	$SE(\hat{\mu}_X)$	$CP(\hat{\mu}_X)$	$SD(\hat{\mu}_Y)$	$SE(\hat{\mu}_Y)$	$CP(\hat{\mu}_Y)$
$\alpha = 2$	$T_i = X_i$ (40%)	100	0.108	0.106	0.941	0.106	0.107	0.950
$\beta = 2$	$T_i = Y_i$ (40%)	200	0.076	0.075	0.940	0.074	0.075	0.947
$\gamma = 3$	$T_i = C_i$ (20%)	300	0.062	0.062	0.948	0.060	0.061	0.949
	$T_i = X_i$ (30%)	100	0.129	0.128	0.939	0.127	0.129	0.952
	$T_i = Y_i$ (30%)	200	0.094	0.090	0.949	0.090	0.089	0.949
	$T_i = C_i$ (40%)	300	0.075	0.074	0.944	0.073	0.073	0.947
	$T_i = X_i$ (20%)	100	0.195	0.178	0.941	0.188	0.179	0.948
	$T_i = Y_i$ (20%)	200	0.127	0.122	0.929	0.121	0.121	0.949
	$T_i = C_i$ (60%)	300	0.100	0.098	0.951	0.097	0.098	0.946

Results

Table 5(b). Simulation results one the MLE of the mean failure time $\mu_X = E(X)$ and $\mu_Y = E(Y)$ based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\mu}_X)$	$SE(\hat{\mu}_X)$	$CP(\hat{\mu}_X)$	$SD(\hat{\mu}_Y)$	$SE(\hat{\mu}_Y)$	$CP(\hat{\mu}_Y)$
$\alpha = 2$	$T_i = X_i$ (59%)	100	0.042	0.042	0.946	0.056	0.058	0.958
$\beta = 4$	$T_i = Y_i$ (21%)	200	0.030	0.030	0.945	0.039	0.041	0.959
$\gamma = 5$	$T_i = C_i$ (20%)	300	0.025	0.024	0.943	0.033	0.033	0.941
	$T_i = X_i$ (45%)	100	0.050	0.048	0.929	0.067	0.068	0.951
	$T_i = Y_i$ (15%)	200	0.035	0.034	0.931	0.047	0.048	0.946
	$T_i = C_i$ (40%)	300	0.029	0.028	0.948	0.038	0.039	0.953
	$T_i = X_i$ (30%)	100	0.063	0.060	0.934	0.085	0.087	0.963
	$T_i = Y_i$ (10%)	200	0.045	0.043	0.931	0.059	0.061	0.954
	$T_i = C_i$ (60%)	300	0.035	0.035	0.946	0.051	0.050	0.943

Results

Table 5(c). Simulation results one the MLE of the mean failure time $\mu_X = E(X)$ and $\mu_Y = E(Y)$ based on 1,000 repetitions.

Par.	Proportion	n	$SD(\hat{\mu}_X)$	$SE(\hat{\mu}_X)$	$CP(\hat{\mu}_X)$	$SD(\hat{\mu}_Y)$	$SE(\hat{\mu}_Y)$	$CP(\hat{\mu}_Y)$
$\alpha=3$	$T_i = X_i$ (29%)	100	0.032	0.032	0.954	0.026	0.027	0.958
$\beta=2$	$T_i = Y_i$ (51%)	200	0.023	0.023	0.939	0.020	0.019	0.938
$\gamma=7$	$T_i = C_i$ (20%)	300	0.018	0.018	0.945	0.016	0.016	0.949
	$T_i = X_i$ (21%)	100	0.038	0.037	0.943	0.033	0.032	0.944
	$T_i = Y_i$ (39%)	200	0.027	0.026	0.952	0.024	0.022	0.933
	$T_i = C_i$ (40%)	300	0.022	0.022	0.955	0.018	0.018	0.955
	$T_i = X_i$ (14%)	100	0.047	0.046	0.939	0.040	0.039	0.944
	$T_i = Y_i$ (26%)	200	0.035	0.033	0.937	0.028	0.027	0.934
	$T_i = C_i$ (60%)	300	0.027	0.027	0.948	0.022	0.022	0.949

Outline

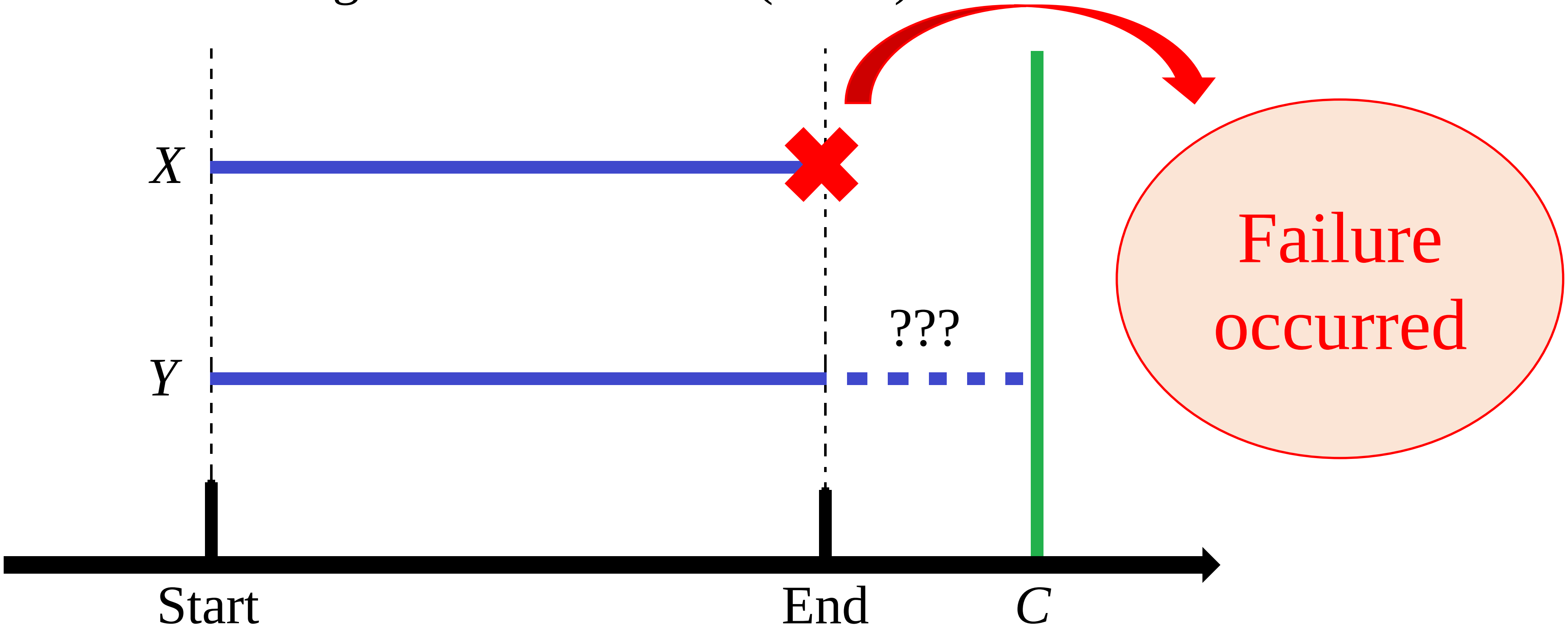
- Introduction
- Background
- Dependence measures under the generalized FGM copula
- Competing risks analysis under the generalized FGM copula model
- Simulation
- **Data analysis**
- Concluding remarks

Radio transmitter-receivers data (Mendenhall and Hader 1958)

X : Confirmed failure time (hours)

Y : Unconfirmed failure time (hours)

C : Censoring time fixed at 630 (hours)



Data analysis

Table 6. The summary of the transmitter-receivers data from Mendenhall and Hader (1958).

	Confirmed ($\delta_i = 1$)	Unconfirmed ($\delta_i^* = 1$)	Censored ($\delta_i = \delta_i^* = 0$)
The number of events (event rate %)	218 (59%)	107 (29%)	44 (12%)
Average (hours)	$\bar{X} = 229.6193$	$\bar{Y} = 191.1963$	$C = 630$ (fixed)

Data analysis

X : Confirmed failure time (hours)

Y : Unconfirmed failure time (hours)

C : Censoring time fixed at 630 (hours)

- We aim to fit the model

$$F_{\varphi}(x, y) = (1 + x^{-\gamma})^{-\alpha}(1 + y^{-\gamma})^{-\beta} \\ \times [1 + \theta\{1 - (1 + x^{-\gamma})^{-\alpha p}\}^{q}\{1 - (1 + y^{-\gamma})^{-\beta p}\}^{q}].$$

Data analysis

- We set $p = 3$ and $q = 2$ to allow wide range of τ and ρ.
- Due to nonidentifiability, one needs to choose dependence parameters by some prior knowledge or speculation.

The radios which were assumed to be failed but finally without confirmed may be treated as the radios having a minor problem.

↓

Y may be associated with the true failure time X.

- We choose Kendall’s tau is equal to 0.3 which corresponds to $\theta = 0.74$.

Results

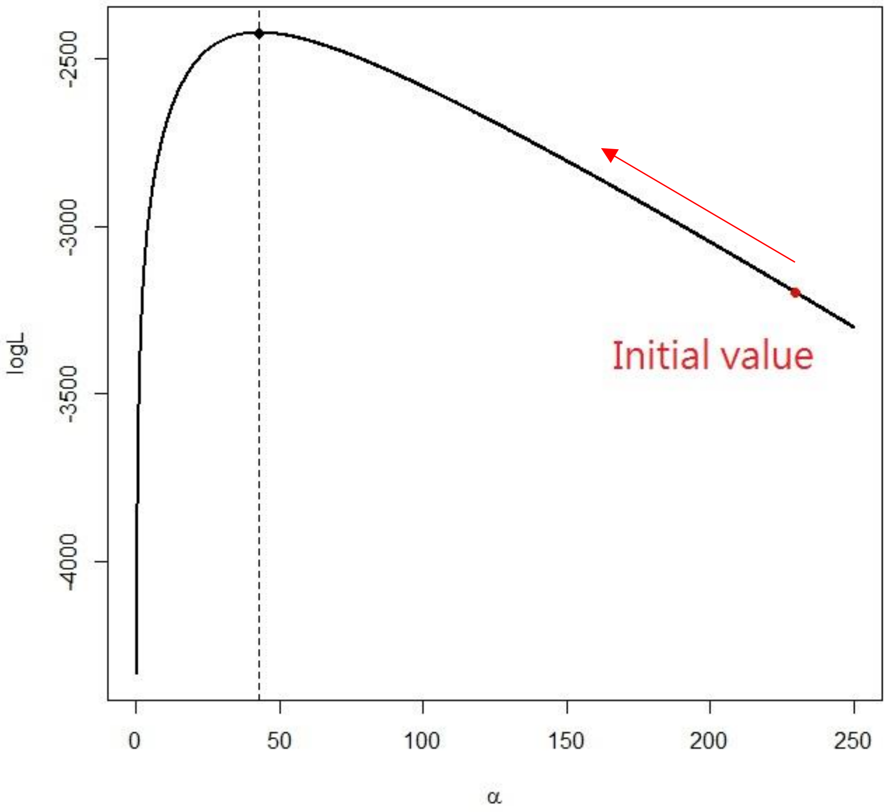
Table 7. Records of iteration steps of Algorithm 1.

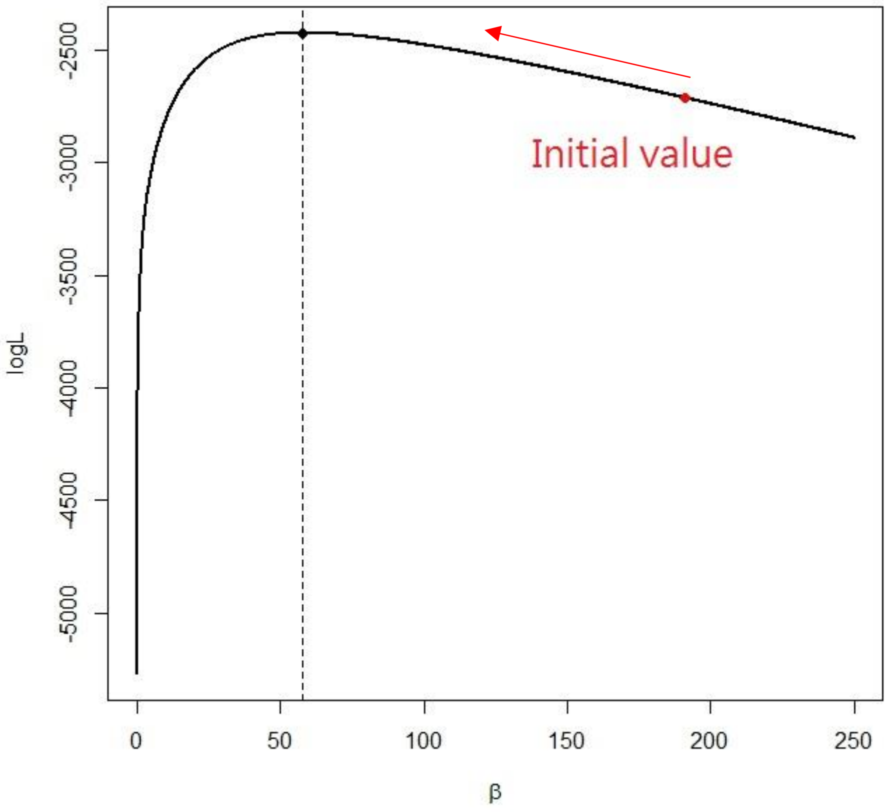
Iteration	$\alpha^{(k)}$	$\beta^{(k)}$	$\gamma^{(k)}$	Log-likelihood	Score vector
$k=0$	229.619	191.196	1	-2553.301	(-286.875, 7.425, 743.289)
$k=1$	31.680	41.054	0.691	-2425.776	(2.499, 27.630,-18.539)
$k=2$	42.473	57.811	0.750	-2421.411	(1.376, 0.297, -5.194)
$k=3$	42.683	57.991	0.750	-2421.407	(-0.002, -0.001, 0.006)
$k=4$	42.682	57.990	0.750	-2421.407	(0, 0, 0)
$k=5$	42.682	57.990	0.750	-2421.407	(0, 0, 0)

Results of the MLE, standard error and 95% confidence interval

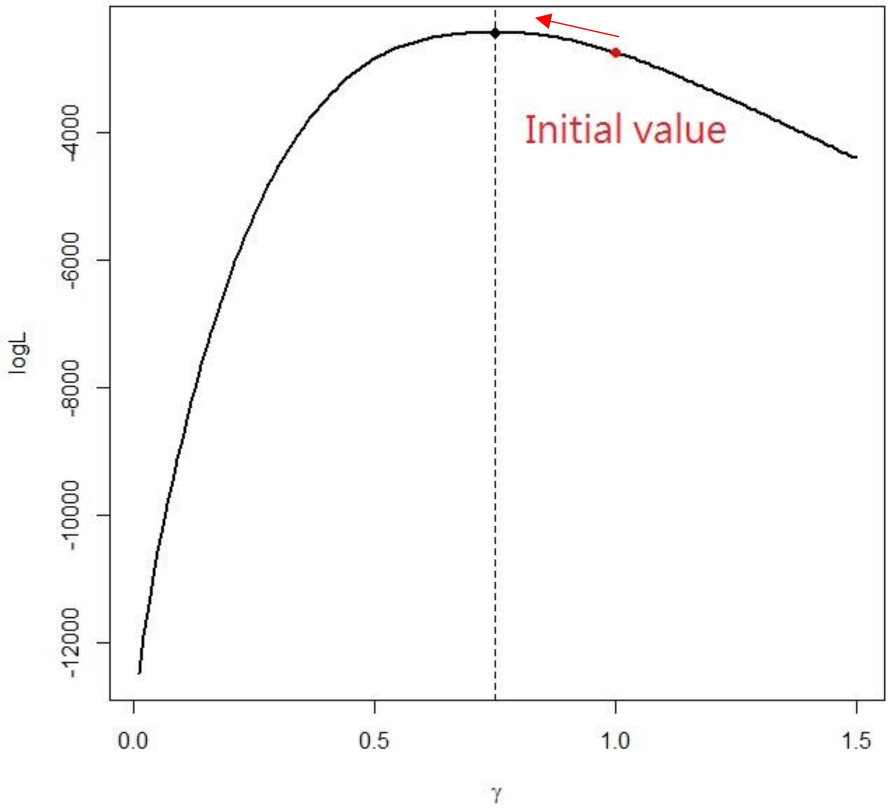
	α	β	γ
MLE	42.68	57.99	0.75
SE	5.09	6.80	0.02
95% CI	(33.78, 53.92)	(46.08, 72.98)	(0.70, 0.80)

Profile plots of log-likelihood

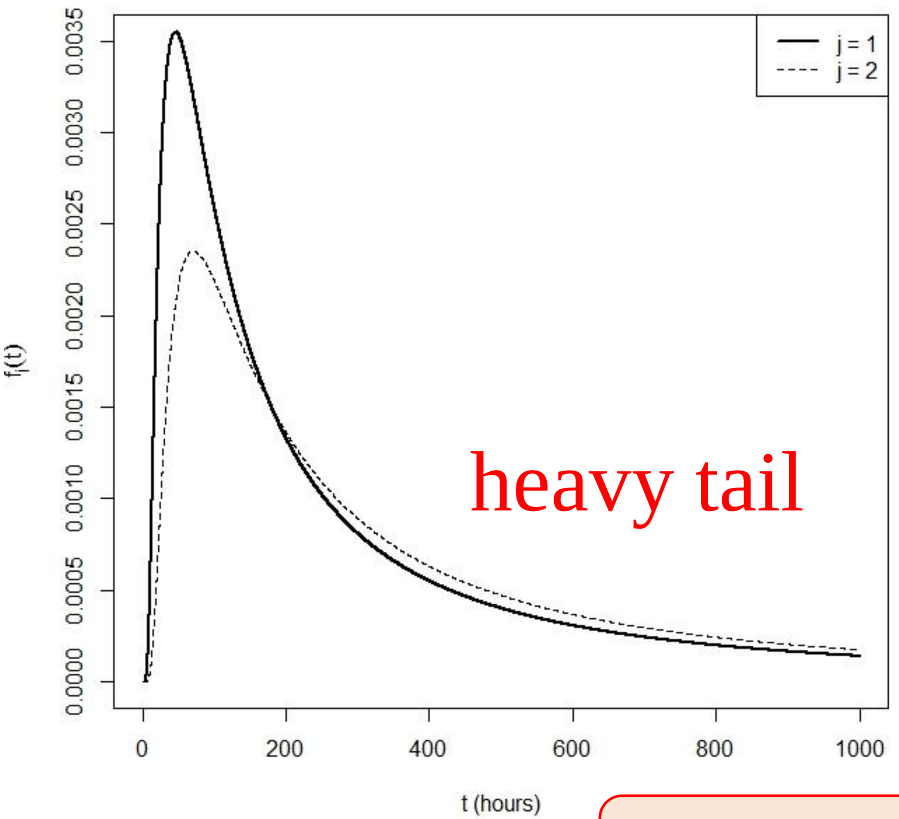


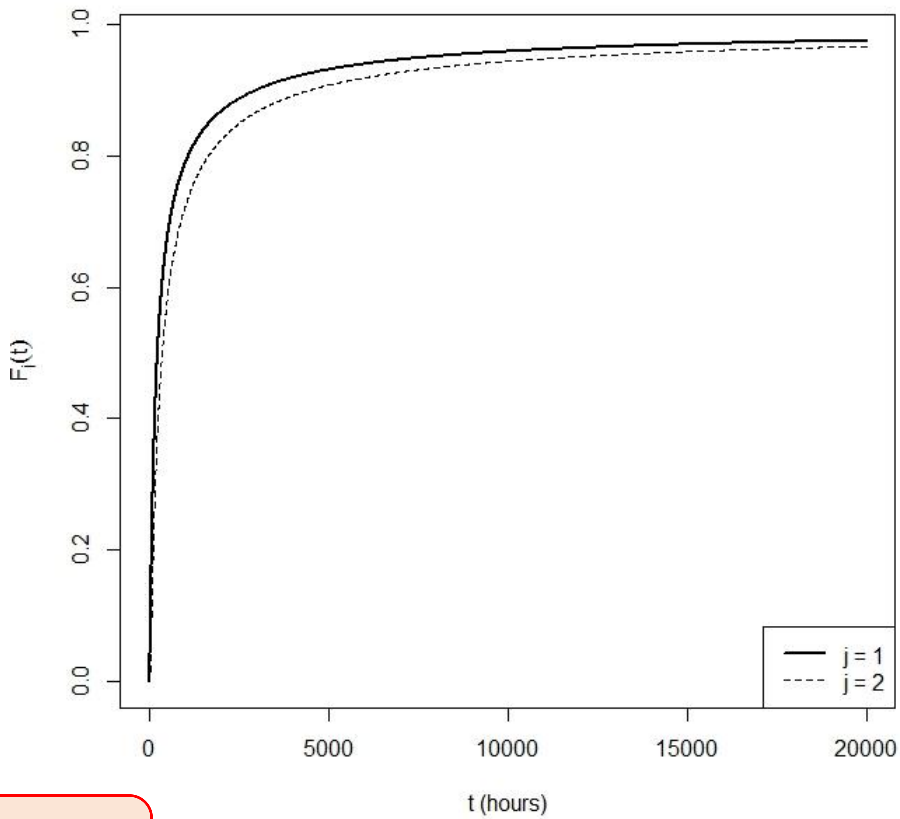


Profile plots of log-likelihood



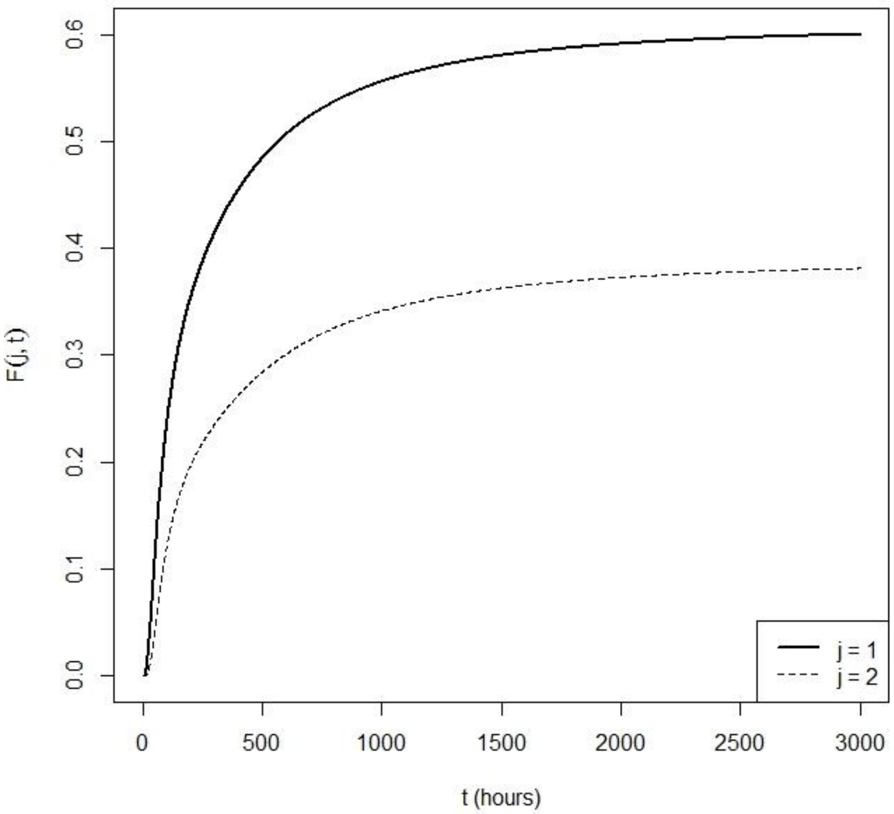
Fitted pdf and cdf





$$\hat{\gamma} = 0.75 < 1$$

Fitted sub-distribution



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Concluding remarks

- New results obtained **(Theorem 1 – 6, Corollary 1).**
- Maximize the log-likelihood function by a randomize Newton-Raphson algorithm **(Algorithm 1).**
- From our results, we think that the generalized FGM copula can be used for routine practice in competing risks data analysis.

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Thank you !