Biom. J. 32 (1990) 1, 47–57 Akademie-Verlag Berlin

A Study of Distribution-free Tests for Umbrella Alternatives

Y. I. Chen and D. A. Wolfe
Department of Statistics, The Ohio State University, USA

Summary

In this paper we are concerned with test procedures for umbrella alternatives in the k-sample location problem. Distribution-free tests are considered for both cases where the peak of the umbrella is known or unknown. Comparative results of a Monte Carlo power study are presented.

Key words: Distribution-free tests; Monte Carlo power study; Ordered alternatives; Umbrella alternatives.

1. Introduction

A problem that occurs frequently in statistical data analyses is to determine whether k sets of independent observations arose from the same population. A variety of nonparametric tests have been developed for this k-sample setting. In particular, Kruskal and Wallis (1952) considered a distribution-free test for general location alternatives to the null hypothesis of one common distribution. Jonckheere (1954) and Terpstra (1952) carried out the initial studies for testing against ordered location alternatives. Chacko (1963) proposed another test for ordered alternatives, which is similar in construction to the one proposed by Kruskal and Wallis for general alternatives. For the case of umbrella alternatives, which include ordered alternatives as a special case, Mack and Wolfe (1981) are the first to provide a general solution to this problem in the k-sample setting. Simpson and Margolin (1986) discussed a recursive procedure for testing an increasing dose-response relationship when a downturn in response at high dose is possible. Hettmansperger and Norton (1987) also considered a general approach to testing for various restricted alternatives.

In this paper we are concerned with umbrella alternatives and consider several competing tests for when the peak of the umbrella is known a priori and for the more common practical setting where the peak of the umbrella is unknown. Such alternatives are appropriate for many problems. For example, an experimenter in psychology usually expects that an increase in stress (or training) produces an increasing negative (or positive) effect on performance of some task. Moreover, it is generally believed that learning ability is an increasing function of age up to a certain point and then it decreases with increasing age. Other examples are in medicine where therapies often become counter-productive at high doses. In such cases, an increasing dose-response relationship with a downturn in response at high doses is anticipated.

In Section 2 we describe the umbrella model under consideration in this paper and discuss previously proposed test procedures for either the peak known or unknown settings. In Section 3 we propose a natural generalization of CHACKO's (1963) statistic to obtain a test for umbrella alternatives when the peak is known a priori. In Section 4 we propose an alternative distribution-free extension of the MACK-WOLFE (1981) statistics to the unknown peak setting. In Section 5 we present the results of an extensive Monte Carlo simulation investigation of the relative powers of these competing distribution-free tests for a variety of umbrella alternative configurations.

2. The Setting, Notation, and Previous Work

Suppose that $X_{i1}, \ldots, X_{in_i}$, $i=1, \ldots, k$, are k independent random samples from populations with continuous distribution functions $F_i(x)=F(x-\vartheta_i)$, $i=1, \ldots, k$. We consider testing the null hypothesis $H_0: [\vartheta_1=\ldots=\vartheta_k]$ against the class of umbrella alternatives $H_A: [\vartheta_1 \leqq \ldots \leqq \vartheta_\alpha \geqq \ldots \geqq \vartheta_k$, for some α, with at least one strict inequality]. In this article, we discuss both the setting where α, the peak of the umbrella, is known and where it is unknown.

Let R_{ij} be the rank of X_{ij} among the $N=\sum_{i=1}^{k} n_i$ observations and let $\bar{R}_i = \sum_{j=1}^{n_i} R_{ij}/n_i$ be the average rank of the ith sample. Set $\lambda_i = n_i/N$, $i=1, \ldots, k$. For testing H_0 against ordered alternatives (corresponding to umbrella with known peak $\alpha=k$), the JONCKHEERE (1954)-TERPSTRA (1952) test rejects for large values of the statistic

$$J=\sum_{i=1}^{k-1} \sum_{j=i+1}^{k} U_{ij}, \tag{2.1}$$

where U_{ij} is the usual Mann-Whitney statistic corresponding to the number of observations in sample j that exceed observations in sample i. MACK and WOLFE (1981) extended this methodology to an arbitrary peak-known (α) umbrella alternative H_A by combining a Jonckheere-Terpstra statistic and a reverse Jonckheere-Terpstra statistic to base their test on rejecting H_0 for large values of

$$A_\alpha=\sum_{i=1}^{\alpha-1} \sum_{j=i+1}^{\alpha} U_{ij}+\sum_{i=\alpha}^{k-1} \sum_{j=i+1}^{k} U_{ji}. \tag{2.2}$$

For the more general unknown peak alternative, Mack and Wolfe proposed to reject H_0 for large values of

$$(2.3) \qquad A_{\hat{\alpha}}^* = \frac{A_{\hat{\alpha}} - \mu_0(A_{\hat{\alpha}})}{\sigma_0(A_{\hat{\alpha}})},$$

where

$$(2.4) \qquad \mu_0(A_t) = \left[N_1^2 + N_2^2 - \sum_{i=1}^{k} n_i^2 - n_t^2\right] \Big/ 4$$

and

$$(2.5) \qquad \sigma_0^2(A_t) = \frac{1}{72}\left\{2\,(N_1^3 + N_2^3) + 3\,(N_1^2 + N_2^2) - \sum_{i=1}^{k} n_i^2\,(2n_i + 3) - n_t^2\,(2n_t + 3) + 12 n_t N_1 N_2 - 12 n_t^2 N\right\},$$

with $N_1 = \sum_{i=1}^{t} n_i$ and $N_2 = \sum_{i=t}^{k} n_i$, are the null (H_0) mean and variance, respectively, of A_t, $t = 1, \ldots, k$, and $\hat{\alpha}$ is a sample estimate of the unknown peak α. (See MACK and WOLFE (1981) for details on their estimator $\hat{\alpha}$.)

An entirely different approach leads to the ordered alternatives test proposed by CHACKO (1963). Let $\hat{R}_1 \leqq \hat{R}_2 \leqq \ldots \leqq \hat{R}_k$ be the isotonic regression of the average ranks $\bar{R}_1, \ldots, \bar{R}_k$ under the order restriction $\vartheta_1 \leqq \ldots \leqq \vartheta_k$. (For a discussion of the algorithm for obtaining $\hat{R}_1, \ldots, \hat{R}_k$, see BARLOW, et al. (1972).) Chacko's rank test then rejects H_0 for large values of

$$(2.6) \qquad \bar{\chi}_{[k]}^2 = \frac{12}{(N+1)} \sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2}\right)^2.$$

In a general approach to constructing tests designed for specific patterned alternatives, HETTMANSPERGER and NORTON (1987) proposed two procedures for testing H_0 against the umbrella alternatives H_A. For the case of known umbrella peak α and equally spaced effects, corresponding to $\vartheta_i = \vartheta_0 + i\vartheta$, for $i = 1, \ldots, \alpha$, and $\vartheta_i = \vartheta_0 + (2\alpha - i)\,\vartheta$, for $i = \alpha + 1, \ldots, k$, they proposed rejecting H_0 for large values of the statistic

$$(2.7) \qquad V_\alpha = \left(\frac{12}{N+1}\right)^{1/2} \frac{\sum_{i=1}^{\alpha} \lambda_i\,(i - \bar{c}_w)\,\bar{R}_i + \sum_{i=\alpha+1}^{k} \lambda_i\,(2\alpha - i - \bar{c}_w)\,\bar{R}_i}{\left[\sum_{i=1}^{\alpha} \lambda_i\,(i - \bar{c}_w)^2 + \sum_{i=\alpha+1}^{k} \lambda_i\,(2\alpha - i - \bar{c}_w)^2\right]^{1/2}},$$

where $\bar{c}_w = \sum_{i=1}^{\alpha} i\lambda_i + \sum_{i=\alpha+1}^{k} (2\alpha - i)\,\lambda_i$. For the same equally spaced alternative and unknown umbrella peak α, they suggested rejecting H_0 for large values of

$$(2.8) \qquad V_{\max}^* = \max_{1 \leqq t \leqq k} V_t,$$

where V_t is given by (2.7) for $t = 1, \ldots, k$.

Finally SIMPSON and MARGOLIN (1986) suggested a recursive procedure for investigating an increasing dose-response relationship when there is potential for a drop in response at high dose levels. Set

$$Q_j = \sum_{i=1}^{j-1} U_{ij} \tag{2.9}$$

for $j=2, \ldots, k$, where the U_{ij}'s are the same Mann-Whitney statistics used in defining J (2.1) and A_α (2.2). Let

$$S_t = \sum_{i=1}^{t-1} \sum_{j=i+1}^{t} U_{ij} \tag{2.10}$$

be the Jonckheere-Terpstra statistic for the first t samples, $t=2, \ldots, k$. Setting $M = \max_{2 \leq j \leq k} \left\{ j: Q_j \geq \frac{(n_1 + \ldots + n_{j-1})\, n_j}{2} \right\}$, the form of the Simpson-Margolin test considered in this paper rejects H_0 for large values of

$$S_M\left(\frac{1}{2}\right) = Q_2 + \ldots + Q_M . \tag{2.11}$$

3. Generalization of Chacko's Test to Umbrella Alternatives With Peak Known

When, under the alternative, the peak (α) of the umbrella is known a priori, Chacko's statistic is generalized to be

$$\bar{\chi}^2_{[\alpha]} = \frac{12}{(N+1)} \sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2}\right)^2 , \tag{3.1}$$

where $\hat{R}_1 \leq \ldots \leq \hat{R}_\alpha \geq \ldots \geq \hat{R}_k$ is the isotonic regression of $\bar{R}_1, \ldots, \bar{R}_k$ with weights $\lambda_1, \ldots, \lambda_k$. Note that the derivation of the $\hat{R}_i$'s is a quadratic programming problem. The object is to minimize

$$\sum_{i=1}^{k} \lambda_i (r_i - \bar{R}_i)^2 , \tag{3.2}$$

subject to the constraints

$$r_1 \leq \ldots \leq r_\alpha \geq \ldots \geq r_k$$

and

$$\sum_{i=1}^{k} \lambda_i r_i = (N+1)/2 . \tag{3.3}$$

However, under umbrella alternatives each location parameter except the one for the peak group has exactly one immediate predecessor. Therefore, an algorithm similar to the Minimum Violation algorithm discussed in BARLOW et al. (1972) can be applied to obtain the isotonic regression $\hat{R}_1 \leq \ldots \leq \hat{R}_\alpha \geq \ldots \geq \hat{R}_k$. This algorithm can be described in the following way: if $\bar{R}_1 \leq \ldots \leq \bar{R}_\alpha \geq \ldots \geq \bar{R}_k$, then

$\hat{R}_i=\bar{R}_i$, $i=1, \ldots, k$; otherwise, we start with the average rank of the peak group, $\bar{R}_\alpha$. We look for violators, where $\bar{R}_i$ is a violator if $\bar{R}_i>\bar{R}_{i+1}$ for $i=1, \ldots, \alpha-1$ or $\bar{R}_i>\bar{R}_{i-1}$ for $i=\alpha+1, \ldots, k$. The algorithm begins by choosing a violator and pooling it with its immediate predecessor to form a weighted average rank. This violator and its immediate predecessor will then be replaced by the weighted average rank. Consequently, the weighted average rank is regarded as the immediate predecessor and is then compared with the adjacent ones and so on. This procedure is continued until a set of quantities satisfying (3.3) is obtained. Note that when we start with $\bar{R}_\alpha$ we may immediately have two adjacent violators. In this case, the average rank which has the maximum value between the two involved averages is assigned to both $\hat{R}_\alpha$ and the adjacent group (either $\hat{R}_{\alpha-1}$ or $\hat{R}_{\alpha+1}$) that leads to this maximum.

Using an argument similar to that of HOGG (1965), HETTMANSPERGER and NORTON (1987) showed that

$$(3.4)\qquad (\bar{\chi}^2_{[k]})^{1/2}=\max\left\{\left(\frac{12}{N+1}\right)^{1/2}\sum_{i=1}^{k} b_i\lambda_i\bar{R}_i\right\},$$

where the maximum is taken over choices of $b_1, \ldots, b_k$ such that $\sum\lambda_i b_i=0$, $\sum\lambda_i b_i^2=1$ and $b_1\leqq\ldots\leqq b_k$. In fact, we now prove, in addition, that, for $\alpha=1, \ldots, k$,

$$(3.5)\qquad (\bar{\chi}^2_{[\alpha]})^{1/2}=\max\left\{\left(\frac{12}{N+1}\right)^{1/2}\sum_{i=1}^{k} c_i\lambda_i\bar{R}_i\right\},$$

where the maximum is now taken over selections of $c_1, \ldots, c_k$ such that $\sum\lambda_i c_i=0$, $\sum\lambda_i c_i^2=1$ and $c_1\leqq\ldots\leqq c_\alpha\geqq\ldots\geqq c_k$.

Proof:

Since $\sum\lambda_i c_i=0$, we can write

$$\sum_{i=1}^{k} c_i\lambda_i\bar{R}_i=\left[\sum_{i=1}^{k}\lambda_i\left(\bar{R}_i-\frac{N+1}{2}\right)^2\right]^{1/2}\sum_{i=1}^{k}\frac{\lambda_i c_i\left(\bar{R}_i-\frac{N+1}{2}\right)}{\left[\sum_{j=1}^{k}\lambda_j\left(\bar{R}_j-\frac{N+1}{2}\right)^2\right]^{1/2}}$$

Let $u=\lambda_i^{1/2}c_i$, $v=\lambda_i^{1/2}\left(\bar{R}_i-\frac{N+1}{2}\right)\Big/\left[\sum_{j=1}^{k}\lambda_j\left(\bar{R}_j-\frac{N+1}{2}\right)^2\right]^{1/2}$. Using the identity $uv=[u^2+v^2-(u-v)^2]/2$, we then have

$$\sum_{i=1}^{k} c_i\lambda_i\bar{R}_i=\frac{1}{2}\left[\sum_{i=1}^{k}\lambda_i\left(\bar{R}_i-\frac{N+1}{2}\right)^2\right]^{1/2}\left\{\sum_{i=1}^{k}\lambda_i c_i^2+\frac{\sum_{i=1}^{k}\lambda_i\left(\bar{R}_i-\frac{N+1}{2}\right)^2}{\sum_{j=1}^{k}\lambda_j\left(\bar{R}_j-\frac{N+1}{2}\right)^2}-\frac{\sum_{i=1}^{k}\lambda_i\left[c_i\left(\sum_{j=1}^{k}\lambda_j\left(\bar{R}_j-\frac{N+1}{2}\right)^2\right)^{1/2}-\left(\bar{R}_i-\frac{N+1}{2}\right)\right]^2}{\sum_{j=1}^{k}\lambda_j\left(\bar{R}_j-\frac{N+1}{2}\right)^2}\right\}$$

Since $\sum \lambda_i c_i^2 = 1$, the above expression is maximized by minimizing

$$\sum_{i=1}^{k} \lambda_i \left[c_i \left(\sum_{j=1}^{k} \lambda_j \left(\hat{R}_j - \frac{N+1}{2} \right)^2 \right)^{1/2} - \left(\bar{R}_i - \frac{N+1}{2} \right) \right]^2 ,$$

under the restriction $c_1 \leqq \ldots \leqq c_\alpha \geqq \ldots \geqq c_k$. However, this minimum can be obtained by selecting the c_i's so that

$$c_i = \frac{\left(\hat{R}_i - \frac{N+1}{2} \right)}{\left[\sum_{j=1}^{k} \lambda_j \left(\hat{R}_j - \frac{N+1}{2} \right)^2 \right]^{1/2}}$$

for $i = 1, \ldots, k$. We then see that

$$\max \sum_{i=1}^{k} c_i \lambda_i \bar{R}_i = \frac{1}{2} \left[\sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2} \right)^2 \right]^{1/2} \times \left\{ 1 + \frac{\sum_{i=1}^{k} \lambda_i \left(\bar{R}_i - \frac{N+1}{2} \right)^2 - \sum_{i=1}^{k} \lambda_i (\hat{R}_i - \bar{R}_i)^2}{\sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2} \right)^2} \right\}$$

Since $\sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2} \right)^2 = \sum_{i=1}^{k} \lambda_i \left(\bar{R}_i - \frac{N+1}{2} \right)^2 - \sum_{i=1}^{k} \lambda_i (\hat{R}_i - \bar{R}_i)^2$, we have

$$\max \sum_{i=1}^{k} c_i \lambda_i \bar{R}_i = \left[\sum_{i=1}^{k} \lambda_i \left(\hat{R}_i - \frac{N+1}{2} \right)^2 \right]^{1/2}$$

and (3.5) holds.

4. Alternative Adaptation of Mack-Wolfe Statistic to Umbrella Alternatives With Peak Unknown

If the peak of the umbrella is unknown, the alternative H_A can be viewed as a union of k individual umbrella alternatives with the peak at group $1, \ldots, k$, respectively; that is, $H_A = \bigcup_{t=1}^{k} H_{At}$, where H_{At} corresponds to $\vartheta_1 \leqq \ldots \leqq \vartheta_{t-1} \leqq \vartheta_t \geqq \geqq \vartheta_{t+1} \geqq \ldots \geqq \vartheta_k$, with at least one strict inequality. This way of viewing H_A leads to a natural extension of the known peak test based on A_α(2.2) to the unknown peak setting that is different from the one based on A_α^*(2.3) and studied by MACK and WOLFE (1981). This natural extension corresponds to rejecting H_0 for large values of

(4.1) $$A_{\max}^* = \max_{1 \leqq t \leqq k} A_t^* ,$$

where $A_t^* = \frac{A_t - \mu_0(A_t)}{\sigma_0(A_t)}$ and A_t, $\mu_0(A_t)$ and $\sigma_0^2(A_t)$ are given in equations (2.2), (2.4) and (2.5), respectively. This test based on $A_{\max}^*$ is similar in form to the Hettmansperger-Norton unknown peak test based on $V_{\max}^*$ (2.8).

5. Monte Carlo Power Study

To examine the relative powers of these competing distribution-free test procedures for general umbrella alternatives, we conducted a Monte Carlo power study. We considered both $k=4$ and $k=5$ populations, with $n_1=\ldots=n_k=3$ observations per sample in each case, and a variety of different umbrella alternatives.

For each of these settings, the International Mathematical and Statistical Libraries (IMSL) routine RNUN was used to generate uniformly distributed random numbers in (0,1]. Routines RNNOR and RNEXP were then employed to generate appropriate normal and exponential deviates according to the pertinent alternative. In each case, we used 10,000 replications in obtaining the various power estimates. Exact critical values were used, when available, in the sample rejection counts; otherwise, simulated critical values were used. The simulated power estimates for the eight tests considered in this paper are presented in Tables 1 and 2. The designated alternative configurations correspond to values of $\vartheta_1, \ldots, \vartheta_k$.

Table 1

Monte Carlo Power Estimates for $k=4$ and $n_1=\ldots=n_4=3$

(a) Normal

Umbrella Alternatives Population				Nominal	Tests							
1	2	3	4	Level	A_x^*	A_{max}^*	V_{max}^*	$S_M\left(\frac{1}{2}\right)$	J	A_x	V_x	$\bar{\chi}^2_{[x]}$
0	.5	1.0	1.5	.10	.402	.430	.453	.597	.677	.677	.676	.642
				.05	.285	.289	.306	.472	.514	.514	.520	.475
				.01	.102	.107	.116	.157	.231	.231	.239	.220
0	.5	1.5	1.5	.10	.481	.527	.576	.683	.721	.721	.728	.706
				.05	.339	.374	.430	.548	.562	.562	.579	.540
				.01	.117	.132	.163	.186	.270	.270	.286	.279
0	.5	1.0	.5	.10	.268	.267	.268	.337	.264	.400	.432	.353
				.05	.165	.171	.172	.221	.146	.267	.294	.229
				.01	.037	.038	.040	.043	.039	.085	.087	.064
0	1.0	1.5	.5	.10	.463	.451	.428	.494	.221	.641	.648	.566
				.05	.320	.326	.298	.345	.113	.491	.493	.416
				.01	.079	.081	.077	.082	.025	.211	.185	.153
.5	1.0	.5	0	.10	.276	.261	.272	.116	.023	.412	.448	.363
				.05	.169	.175	.175	.032	.009	.268	.295	.219
				.01	.035	.036	.040	.004	.001	.091	.092	.008
0	1.0	.5	0	.10	.326	.306	.274	.212	.058	.498	.437	.423
				.05	.209	.211	.164	.078	.026	.344	.285	.271
				.01	.043	.044	.033	.012	.005	.125	.084	.099
1.5	1.0	.5	0	.10	.413	.391	.465	.017	.001	.690	.697	.649
				.05	.290	.297	.317	.002	.000	.529	.538	.486
				.01	.101	.106	.115	.000	.000	.237	.236	.240
1.5	1.5	.5	0	.10	.488	.489	.585	.048	.001	.737	.749	.713
				.05	.345	.383	.438	.003	.000	.580	.597	.555
				.01	.123	.136	.171	.000	.000	.267	.281	.300

Table 1

(b) Exponential

Umbrella Alternatives Population 1	2	3	4	Nominal Level	A_α^*	A_{max}^*	V_{max}^*	$S_M\left(\frac{1}{2}\right)$	J	A_α	V_α	$\bar{\chi}_{[\alpha]}^2$
0	.5	1.0	1.5	.10	.592	.619	.611	.757	.816	.816	.811	.772
				.05	.473	.479	.481	.659	.676	.676	.685	.636
				.01	.229	.265	.245	.329	.436	.436	.421	.390
0	.5	1.5	1.5	.10	.605	.664	.683	.770	.828	.828	.829	.807
				.05	.467	.530	.573	.673	.708	.708	.708	.688
				.01	.215	.258	.308	.325	.434	.434	.450	.455
0	.5	1.0	.5	.10	.400	.402	.414	.460	.367	.581	.605	.496
				.05	.263	.277	.297	.335	.235	.417	.464	.356
				.01	.068	.076	.099	.084	.084	.148	.185	.131
0	1.0	1.5	.5	.10	.627	.619	.598	.630	.284	.800	.776	.725
				.05	.472	.483	.469	.495	.164	.668	.659	.586
				.01	.149	.155	.172	.171	.046	.338	.344	.292
.5	1.0	.5	0	.10	.397	.381	.408	.155	.019	.569	.606	.496
				.05	.262	.276	.297	.044	.008	.407	.451	.351
				.01	.067	.076	.097	.008	.001	.145	.193	.152
0	1.0	.5	0	.10	.460	.440	.388	.256	.061	.663	.569	.585
				.05	.311	.318	.256	.107	.028	.498	.401	.409
				.01	.077	.079	.056	.018	.005	.200	.143	.184
1.5	1.0	.5	0	.10	.581	.561	.600	.011	.000	.830	.814	.762
				.05	.463	.473	.472	.000	.000	.706	.684	.627
				.01	.235	.245	.248	.000	.000	.418	.403	.400
1.5	1.5	.5	0	.10	.604	.619	.681	.049	.000	.829	.833	.801
				.05	.465	.522	.569	.002	.000	.707	.711	.688
				.01	.209	.258	.302	.000	.000	.421	.429	.454

The simulation results suggest several conclusions. The Jonckheere-Terpstra test, J, is generally better than Chacko's test, $\bar{\chi}_{[k]}^2$, for ordered alternatives. In the peak known setting, both V_α and A_α are superior to $\bar{\chi}_{[\alpha]}^2$ against umbrella alternatives. For $1<\alpha<k$, V_α provides a better test than does A_α for equal spacing alternatives. However, when the alternatives are not equally spaced, the test V_α may not be as powerful as A_α, especially for exponential data. For the unknown peak setting, the recursive test $S_M\left(\frac{1}{2}\right)$ has much higher power than the other tests considered here for the settings where the peak group is relatively close to the k^{th} population. When, however, the location of the peak group is relatively far from the k^{th} population, the recursive test performs poorly. In these cases, the three tests based on A_{max}^*, A_α^* and V_{max}^*, respectively, all do better than the one based on $S_M\left(\frac{1}{2}\right)$.

Finally, it seems natural to consider development of a peak unknown analogue of the test based on the umbrella alternatives version, $\bar{\chi}^2_{[\alpha]}$, of Chacko's statistic. However, in view of the relative performances of the tests based on A_α, V_α or $\bar{\chi}^2_{[\alpha]}$ it seems doubtful that such a test would do any better than the available procedures based on $A^*_{\hat{\alpha}}$, $A^*_{\max}$, $V^*_{\max}$, or $S_M\left(\frac{1}{2}\right)$

Table 2

Monte Carlo Power Estimates for $k=5$ and $n_1=\ldots=n_5=3$

(a) Normal

Umbrella Alternatives Population 1	2	3	4	5	Nominal Level	Tests $A^*_{\hat{\alpha}}$	$A^*_{\max}$	$V^*_{\max}$	$S_M\left(\frac{1}{2}\right)$	J	A_α	V_α	$\bar{\chi}^2_{[\alpha]}$
0	.5	1.0	1.5	2.0	.10	.653	.697	.731	.821	.897	.897	.896	.862
					.05	.523	.558	.588	.702	.799	.799	.805	.743
					.01	.267	.281	.302	.405	.476	.476	.501	.447
0	0	1.0	1.5	1.5	.10	.562	.638	.691	.761	.834	.834	.841	.811
					.05	.432	.489	.544	.622	.707	.707	.720	.676
					.01	.180	.201	.263	.308	.360	.360	.403	.387
0	.5	1.0	1.5	1.0	.10	.461	.496	.521	.595	.576	.648	.716	.598
					.05	.317	.347	.356	.437	.413	.488	.564	.435
					.01	.110	.111	.139	.156	.139	.194	.230	.167
0	.5	1.0	1.5	0	.10	.551	.556	.476	.561	.174	.759	.579	.688
					.05	.394	.397	.298	.421	.085	.608	.410	.530
					.01	.136	.118	.102	.144	.015	.275	.117	.232
0	1.0	2.0	1.0	0	.10	.776	.763	.780	.616	.054	.901	.917	.835
					.05	.623	.598	.624	.442	.021	.807	.833	.697
					.01	.297	.292	.350	.141	.001	.526	.553	.369
0	.5	2.0	1.0	.5	.10	.645	.639	.635	.535	.221	.827	.802	.758
					.05	.470	.445	.442	.362	.117	.697	.660	.599
					.01	.173	.171	.182	.100	.017	.382	.325	.255
1.0	1.5	1.0	.5	0	.10	.464	.498	.524	.106	.003	.643	.718	.603
					.05	.321	.350	.356	.025	.001	.498	.565	.429
					.01	.108	.108	.137	.004	.000	.215	.251	.171
.5	2.0	1.0	.5	0	.10	.648	.664	.590	.263	.004	.868	.799	.782
					.05	.514	.526	.412	.041	.001	.762	.653	.632
					.01	.209	.185	.159	.005	.000	.459	.312	.299
2.0	1.5	1.0	.5	0	.10	.655	.698	.729	.016	.000	.896	.903	.870
					.05	.526	.563	.589	.002	.000	.801	.808	.757
					.01	.262	.281	.300	.000	.000	.541	.532	.494
1.5	1.5	1.0	.5	0	.10	.523	.576	.615	.049	.001	.773	.784	.743
					.05	.382	.423	.459	.008	.000	.635	.640	.602
					.01	.150	.163	.201	.001	.000	.346	.345	.328

b) Exponential

Umbrella Alternatives Population 1	2	3	4	5	Nominal Level	Tests $A^*_{\hat{\alpha}}$	$A^*_{\max}$	$V^*_{\max}$	$S_M\left(\frac{1}{2}\right)$	J	A_α	V_α	$\bar{\chi}^2_{[\alpha]}$
0	.5	1.0	1.5	2.0	.10	.806	.839	.823	.915	.962	.962	.950	.921
					.05	.721	.752	.728	.851	.913	.913	.892	.843
					.01	.486	.513	.494	.637	.704	.704	.678	.629

Table 2 continued

Umbrella Alternatives Population 1	2	3	4	5	Nominal Level	Test A_i^*	A^*_{max}	V^*_{max}	$S_M\left(\frac{1}{2}\right)$	J	A_x	V_x	$\chi^2_{[x]}$
0	0	1.0	1.5	1.5	.10	.683	.759	.777	.847	.913	.913	.913	.883
					.05	.564	.637	.669	.742	.832	.832	.829	.792
					.01	.305	.352	.428	.464	.534	.534	.569	.549
0	.5	1.0	1.5	1.0	.10	.635	.678	.675	.719	.714	.819	.835	.734
					.05	.498	.536	.540	.580	.574	.688	.732	.612
					.01	.227	.243	.300	.275	.278	.357	.439	.338
0	.5	1.0	1.5	0	.10	.713	.722	.625	.738	.245	.886	.692	.826
					.05	.579	.585	.451	.612	.144	.785	.536	.703
					.01	.273	.244	.187	.294	.035	.468	.196	.405
0	1.0	2.0	1.0	0	.10	.885	.880	.886	.720	.072	.969	.963	.907
					.05	.777	.758	.769	.546	.033	.916	.914	.811
					.01	.472	.474	.532	.198	.004	.711	.714	.539
0	.5	2.0	1.0	.5	.10	.780	.778	.759	.617	.292	.934	.898	.862
					.05	.616	.605	.594	.431	.164	.845	.791	.728
					.01	.290	.295	.304	.123	.030	.550	.464	.418
1.0	1.5	1.0	.5	0	.10	.637	.679	.675	.133	.001	.819	.839	.738
					.05	.503	.543	.541	.037	.001	.693	.728	.605
					.01	.235	.245	.300	.007	.000	.395	.461	.340
.5	2.0	1.0	.5	0	.10	.807	.817	.718	.276	.002	.952	.879	.878
					.05	.696	.707	.569	.047	.001	.890	.772	.755
					.01	.379	.349	.298	.005	.000	.651	.486	.466
2.0	1.5	1.0	.5	0	.10	.807	.844	.825	.012	.000	.960	.950	.924
					.05	.717	.750	.728	.001	.000	.911	.894	.852
					.01	.480	.511	.494	.000	.000	.746	.699	.664
1.5	1.5	1.0	.5	0	.10	.679	.733	.739	.047	.000	.886	.882	.838
					.05	.563	.617	.621	.007	.000	.794	.783	.751
					.01	.303	.342	.382	.001	.000	.557	.532	.524

References

BARLOW, R. F., BARTHOLOMEW, D. D., BREMNER, J. M., and BRUNK, H. D., 1972: *Statistical Inference Under Order Restrictions*. New York: John Wiley.

CHACKO, V. J., 1963: Testing homogeneity against ordered alternatives. Ann. Math. Statist. **34**, 945—956.

HETTMANSPERGER, T. P. and NORTON, R. M., 1987: Tests for patterned alternatives in k-sample problems. J. Amer. Statist. Assoc. **82**, 292—299.

HOGG, R. V., 1965: On models and hypotheses with restricted alternatives. J. Amer. Statist. Assoc. **60**, 1153—1162.

JONCKHEERE, A. R., 1954: A distribution-free k-sample test against ordered alternatives. Biometrika **41**, 133—145.

KRUSKAL, W. H. and WALLIS, W. A., 1952: Use of ranks in one-criterion variance analysis. J. Amer. Statist. Assoc. **47**, 583—621.

MACK, G. A. and WOLFE, D. A., 1981: K-sample rank tests for umbrella alternatives. J. Amer. Statist. Assoc. **76**, 175—181.

SIMPSON, D. G. and MARGOLIN, B. H., 1986: Recursive nonparametric testing for dose-response relationships subject to downturns at high doses. Biometrika **73**, 589—596.

TERPSTRA, T. J., 1952: The asymptotic normality and consistency of Kendall's test against trend when ties are present. Indagationes Math. 14, 327–333.

Received Nov. 1988

CHEN, Yuh-Ing or WOLFE, Douglas A.
Department of Statistics
141 Cockins Hall
Ohio State University
1958 Neil Avenue
Columbus, Ohio 43210-1247
U.S.A.