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Robust Umbrella Tests for a Generalized Behrens-Fisher Problem

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Summary

This paper is concerned with testing for umbrella alternatives in a k-sample location problem when the underlying populations have possibly different shapes. Following CHEN and WOLFE (1990b), rank-based modifications of the HETTMANSPERGER-NORTON (1987) tests are considered for both the settings where the peak of the umbrella is known and where it is unknown. The proposed procedures are exactly distribution-free when the continuous populations are identical with any shape. Moreover, the modified test for peak-known umbrella alternatives remains asymptotically distribution-free when the continuous populations are assumed to be symmetric, even if they differ in shapes. Comparative results of a Monte Carlo study are presented.

Key words Distribution-free test; Generalized Behrens-Fisher problem; Hettmansperger-Norton tests; Monte Carlo study; Umbrella alternatives.

1. Introduction

Suppose that $X_{i1}, \ldots, X_{in_i}$, $i=1, \ldots, k$, are k independent random samples from populations with continuous distribution functions $F_1(x), \ldots, F_k(x)$, respectively. For each $i=1, \ldots, k$, let θ_i be the unique median of the ith population. In this paper we consider the problem of testing the null hypothesis $H_0: [\theta_1 = \cdots = \theta_k]$ against the umbrella alternatives $H_A: [\theta_1 \leqq \cdots \leqq \theta_p \geqq \cdots \geqq \theta_k$ for some p, with at least one strict inequality] without making the assumption that the k populations have the same shape. Since the location parameters are of interest, while the populations have possibly different shapes, this problem is usually referred to as a generalized Behrens-Fisher problem.

A variety of nonparametric tests have been developed for umbrella alternatives in a k-sample location problem. In particular, MACK and WOLFE (1981) first provided a general solution to this problem. HETTMANSPERGER and NORTON (1987) also considered a general approach to testing for various restricted alternatives. Note that these nonparametric tests are distribution-free when the underlying populations are identical. However, the levels of these tests will not necessarily be preserved when the populations have the same median but different shapes or scale parameters. Therefore, test procedures which maintain the designated level for this more general problem are needed.

CHEN and WOLFE (1990b) suggested modifications of the Mack-Wolfe umbrella tests for this generalized Behrens-Fisher setting. However, motivated by the results of CHEN (1990) and CHEN and WOLFE (1990a) that the Hettmansperger-Norton tests perform better than do the corresponding Mack-Wolfe tests for equal spacing umbrella pattern location parameters, we consider, in this article, rank-based modifications of the Hettmansperger-Norton tests for the generalized Behrens-Fisher problem. The proposed procedures are exactly distribution-free when the continuous populations are identical with any shape. Moreover, the modified test for peak-known umbrella alternatives remains asymptotically distribution-free when the continuous populations are assumed to be symmetric, even if they differ in shapes.

In Section 2 we review the Hettmansperger-Norton umbrella tests for both the settings where the peak of the umbrella is known and where it is unknown. In Section 3 we modify the Hettmansperger-Norton statistics to obtain rank tests for the generalized Behrens-Fisher problem. In Section 4 we present and discuss the results of a small sample Monte Carlo level and power study.

2. Hettmansperger-Norton Tests

In a general approach to constructing tests designed for specific patterned alternatives, HETTMANSPERGER and NORTON (1987) proposed procedures for testing H_0 against the umbrella alternatives H_A. Let R_{ij} be the rank of X_{ij} among the $N=\sum_{i=1}^{k} n_i$ oberservations and let $\bar{R}_i=\sum_{j=1}^{n_i} R_{ij}/n_i$ be the average rank of the ith sample. Set $\lambda_{Ni}=n_i/N$, $i=1,\ldots,k$. For the case of known umbrella peak p and equally spaced effects, corresponding to $\theta_i=\theta_0+c_{pi}\theta$ with $c_{pi}=i$, for $i=1,\ldots,p$, and $c_{pi}=2p-i$, for $i=p+1,\ldots,k$, they proposed rejecting H_0 for large values of the statistic

$$V_p=\sum_{i=1}^{k} \lambda_{Ni}(c_{pi}-\bar{c}_{pw})\,\bar{R}_i, \tag{2.1}$$

where $\bar{c}_{pw}=\sum_{i=1}^{k} \lambda_{Ni}c_{pi}$. Suppose that $N\rightarrow\infty$ in such a way that $\lambda_{Ni}\rightarrow\lambda_i$, with $0<\lambda_i<1$, $i=1,\ldots,k$. Hettmansperger and Norton also noted that, under H_0, the statistic

$$V_p^*=V_p/\sigma_0(V_p) \tag{2.2}$$

has a limiting ($N\rightarrow\infty$) distribution that is standard normal, where

$$\sigma_0^2(V_p)=\{(N+1)/12\}\sum_{i=1}^{k} \lambda_{Ni}(c_{pi}-\bar{c}_{pw})^2\,. \tag{2.3}$$

For the same equally spaced alternative and unknown umbrella peak, they suggested rejecting H_0 for large values of

$$V^*_{\max} = \underset{1 \leqq t \leqq k}{\text{maximum}} \{V_t^*\}, \tag{2.4}$$

where V_t^* is given by (2.2) for $t = 1, \ldots, k$.

3. Modifications of Hettmansperger-Norton Tests

When the underlying populations are symmetric, the problem of interest in this paper is in fact to test the null hypothesis $H_0^* : (\pi_{ji} = 1/2$ for all pairs of i and $j)$ against the class of alternatives $H_A^* : (\pi_{ji} \leqq 1/2,\ 1 \leqq i < j \leqq p$, and $\pi_{ji} \geqq 1/2,\ p \leqq i < j \leqq k$, for some p, with a least one strict inequality), where $\pi_{ji} = Pr(X_{i1} \geqq X_{j1}) = \int F_j \, dF_i$, $i \neq j = 1, \ldots, k$. Let U_{ji} be the usual Mann-Whitney statistic (MANN and WHITNEY, 1947) corresponding to the number of observations in sample i that exceed observations in sample j. Set $Z_i = \sum_{j \neq i} U_{ji}$ for $i = 1, \ldots, k$. Since $n_i \bar{R}_i = Z_i + n_i(n_i + 1)/2$, $i = 1, \ldots, k$, and $\sum_{i=1}^{k} n_i(c_{pi} - \bar{c}_{pw}) = 0$, the statistic V_p in equation (2.1) can be expressed as

$$V_p = N^{-1} \sum_{i=1}^{k} (c_{pi} - \bar{c}_{pw}) \{Z_i - n_i(N - n_i)/2\}.$$

The expected value of V_p is then given by

$$\mu(V_p) = N^{-1} \sum_{i=1}^{k} (c_{pi} - \bar{c}_{pw}) \{\sum_{j \neq i} n_i n_j (\pi_{ji} - 1/2)\}.$$

It is obvious that the expected value of V_p remains zero under H_0^*. However, the variance of V_p is not constant even under H_0^* when the underlying populations have different shapes. To modify the Hettmansperger-Norton statistics for testing umbrella location alternatives with fewer assumptions on the shapes of the populations, we follow CHEN and WOLFE (1990b) and find, in the following, the variances of V_p, $p = 1, \ldots, k$, under a general setting.

Let

$$\phi_{sti} = \int F_s F_t \, dF_i - \pi_{si} \pi_{ti}, \quad s, t, i = 1, \ldots, k.$$

From BIRNBAUM and KLOSE (1957), we have, for $i \neq j = 1, \ldots, k$,

$$\sigma^2(U_{ji}) = n_i n_j \{(n_j - 1)\, \phi_{jji} + (n_i - 1)\, \phi_{iij} + \pi_{ij} \pi_{ji}\} \tag{3.1}$$

From CHEN and WOLFE (1990b), we also have

$$\operatorname{cov}(U_{ji}, U_{rs}) = \begin{cases} n_i n_j n_s \phi_{isj} & \text{for } j=r,\ i \neq s \\ n_i n_j n_r \phi_{jri} & \text{for } j \neq r,\ i=s \\ -n_i n_j n_s \phi_{jsi} & \text{for } i=r,\ j \neq s \\ -n_i n_j n_r \phi_{irj} & \text{for } i \neq r,\ j=s \\ 0 & \text{if } i, j, r, s \text{ are distinct,} \end{cases} \tag{3.2}$$

and

$$\sigma^2(Z_i) = \sum_{s \neq i} n_i n_s \{(n_i - 1)\, \phi_{iis} + (n_s - 1)\, \phi_{ssi} + \pi_{is}\pi_{si}\} + 2 \sum\sum_{s<t, s \neq i, t \neq i} n_i n_s n_t \phi_{sti},\ i = 1, \ldots, k. \tag{3.3}$$

Using the results in (3.1) and (3.2) we obtain, after some algebraic manipulations, that, for $i<j=2, \ldots, k$,

$$\operatorname{cov}(Z_i, Z_j) = \sum_{s \neq i, s \neq j} n_i n_j n_s (\phi_{ijs} - \phi_{jsi} - \phi_{isj}) - n_i n_j \{(n_j - 1)\, \phi_{jji} + (n_i - 1)\, \phi_{iij} + \pi_{ij}\pi_{ji}\}. \tag{3.4}$$

Therefore, we have

$$\sigma^2(V_p) = N^{-2} \left\{ \sum_{i=1}^{k} (c_{pi} - \bar{c}_{pw})^2\, \sigma^2(Z_i) + 2 \sum\sum_{i<j} (c_{pi} - \bar{c}_{pw})\, (c_{pj} - \bar{c}_{pw}) \operatorname{cov}(Z_i, Z_j) \right\}, \tag{3.5}$$

where $\sigma^2(Z_i)$ and $\operatorname{cov}(Z_i, Z_j)$ are given in equations (3.3) and (3.4), respectively.

Following the suggestions of FLIGNER and POLICELLO (1981), CHEN and WOLFE (1990b) replaced the F_i's with their sample distribution function analogues $\hat{F}_i$'s in the π_{ij} and ϕ_{sti}, respectively, to obtain the consistent estimators

$$\hat{\pi}_{ij} = \bar{P}_{ij}/n_i$$

and

$$\hat{\phi}_{sti} = \int \hat{F}_s \hat{F}_t \, d\hat{F}_i - \hat{\pi}_{si}\hat{\pi}_{ti} = \sum_{v=1}^{n_i} (P_{si}^v - \bar{P}_{si})\, (P_{ti}^v - \bar{P}_{ti})/n_i n_s n_t,$$

where

$$P_{ij}^v = n_i \hat{F}_i(X_{jv}) = \sum_{v=1}^{n_i} \Psi(X_{jv} - X_{iu}), \quad v = 1, \ldots, n_j,$$

and

$$\bar{P}_{ij} = \sum_{v=1}^{n_j} P_{ij}^v / n_j,$$

with

$$\psi(a)=\begin{cases}1 & \text{for } a>0\\ 0 & \text{for } a\leqq 0.\end{cases}$$

By replacing the involved π_{ij}'s and ϕ_{sti}'s with the $\hat{\pi}_{ij}$'s and $\hat{\phi}_{sti}$'s, respectively, in the $N^{-1}\sigma^2(V_p)$, $p=1,\ldots,k$, we obtain the corresponding consistent estimators. To simplify the computation of the estimators, however, we set

$$w_{sti}=\sum_{v=1}^{n_i}(P_{si}^v-\bar{P}_{si})(P_{ti}^v-\bar{P}_{ti}),\ s,\ t,\ i=1,\ldots,k,$$

and replace the (n_i-1)'s by n_i's. (These changes will not affect the asymptotic properties of the estimators.) The estimators of $\sigma^2(Z_i)$ and $\operatorname{cov}(Z_i, Z_j)$ are then respectively obtained as

$$\hat{\sigma}^2(Z_i)=\sum_{s\neq i}(w_{iis}+w_{ssi}+\bar{P}_{is}\bar{P}_{si})+2\sum\sum_{s<t,s\neq i,t\neq i}w_{sti} \tag{3.6}$$

and

$$\hat{\operatorname{cov}}(Z_i, Z_j)=\sum_{s\neq i,s\neq j}(w_{ijs}-w_{jsi}-w_{isj})-(w_{iij}+w_{jji}+\bar{P}_{ij}P_{ji}). \tag{3.7}$$

Therefore, the estimator of $\sigma^2(V_p)$ is given by

$$\hat{\sigma}^2(V_p)=N^{-2}\left\{\sum_{i=1}^{k}(c_{pi}-\bar{c}_{pw})^2\,\hat{\sigma}^2(Z_i)+2\sum\sum_{i<j}(c_{pi}-\bar{c}_{pw})(c_{pj}-\bar{c}_{pw})\,\hat{\operatorname{cov}}(Z_i, Z_j)\right\}, \tag{3.8}$$

where $\hat{\sigma}^2(Z_i)$ and $\hat{\operatorname{cov}}(Z_i, Z_j)$ are given in equations (3.6) and (3.7), respectively. Consequently, for the case of known umbrella peak p and equally spaced effects, we propose rejecting H_0^* in favor of the peak-known (p) umbrella alternative H_A^* for large values of

$$\hat{V}_p^*=V_p/\hat{\sigma}(V_p), \tag{3.9}$$

where V_p and $\hat{\sigma}^2(V_p)$ are given in equations (2.1) and (3.8), respectively. For the same equally spaced umbrella alternative with unknown peak p, we then suggest rejecting H_0^* for large values of

$$\hat{V}_{\max}^*=\underset{1\leqq t\leqq k}{\operatorname{maximum}}\{\hat{V}_t^*\}, \tag{3.10}$$

where $\hat{V}_t^*$ is given by (3.9) for $p=1,\ldots,k$.

Note that the tests based on $\hat{V}_p^*$ and $\hat{V}_{\max}^*$ are both exactly distribution-free when the populations are identical since the $\hat{\sigma}^2(V_t)$'s involve ranks only. Furthermore, suppose that $N \to \infty$ in such a way that $n_i/N \to \lambda_i$, with $0<\lambda_i<1$, $i=1,\ldots,k$. From the results of HETTMANSPERGER and NORTON (1987), we observe that the random variable $V_p/\sigma(V_p)$ has an asymptotic $(N \to \infty)$ null (H_0^*) distribution that is standard normal, where V_p and $\sigma^2(V_p)$ are given in equations (2.1) and (3.5), respectively. It follows from the Glivenko-Cantelli Theorem (see, for example, Theorem 2.1.4A of SERFLING (1980)) that $\hat{F}_i$ converges uniformly to F_i with probability one for $i=1,\ldots,k$. Using this result we obtain that $\sigma^2(V_p)/\hat{\sigma}^2(V_p)$ converges to one almost surely as $N \to \infty$. This implies that the statistic $\hat{V}_p^*$ (3.9) has an asymptotic $(N \to \infty)$ null (H_0^*) distribution that is standard normal. Therefore, we observe that the test based on $\hat{V}_p^*$ is asymptotically distribution-free under H_0^*.

4. Monte Carlo Study

4.1 *Description of the Study*

To investigate the level and power performances of the tests based on the modified Hettmansperger-Norton statistics $\hat{V}_p^*$ (3.9) and $\hat{V}_{\max}^*$ (3.10) relative to those based on the original Hettmansperger-Norton statistics V_p^* and $V_{\max}^*$ given in (2.2) and (2.4), respectively, we conducted a Monte Carlo study. Three families of distributions were selected for these simulations: the normal, contaminated normal and Cauchy. The International Mathematical and Statistical Libraries (IMSL) routines RNNOR and RNCHY were employed to generated appropriate normal and Cauchy deviates. The contaminated normal distribution utilized was a mixture of the standard normal distribution and a normal distribution having mean zero and standard deviation 5 in proportions 0.9 and 0.1, respectively.

In studying the effect of different scale parameters on the significance levels of the test procedures, we considered distribution functions $F_i(x)=F(x/\sigma_i)$, $i=1,\ldots,k$, with $F(0)=1/2$, for several choices of $\sigma_2/\sigma_1,\ldots,\sigma_k/\sigma_1$ and F being normal, contaminated normal or Cauchy. Since the level performance of the test based on $\hat{V}_p^*$ relative to that of the test based on V_p^* is similar for $p=1,\ldots,k$, we simply considered the case $p=k$ in this study. The estimated levels are presented in Table 1.

The power study, reported in Table 2, was designed to compare the powers of the modified test with the original tests for testing against umbrella location alternatives. For the power study to be meaningful, the original tests must maintain their levels. Therefore, we required the distributions to have the same shape. Specifically, we considered $F_i(x)=F(x-\theta_i)$, $i=1,\ldots,k$, with $\theta_1 \leqq \cdots \leqq \theta_p \geqq \cdots \geqq \theta_k$. Several choices of $\theta_2-\theta_1,\ldots,\theta_k-\theta_1$ in combination with the three distributions stated above were studied.

Furthermore, to compare the power performances of the modified Hettmansperger-Norton tests based on $\hat{V}_p^*$ and $\hat{V}_{\max}^*$ with the corresponding modifications of Mack-Wolfe test and Chen-Wolfe test based on $\hat{A}_p^*$ and $\hat{A}_{\max}^*$ for the more general Behrens-Fisher setting, we considered $F_i(x)=F((x-\theta_i)/\sigma_i)$, $i=1,\ldots,k$, for several choices of $\sigma_2/\sigma_1,\ldots,\sigma_k/\sigma_1$ and $\theta_2-\theta_1,\ldots,\theta_k-\theta_1$, and F being normal, contaminated normal or Cauchy. The simulation results are presented in Table 4 and the relating level estimates are reported in Table 3 as a reference.

The level and power studies described above were conducted for $k=4$ populations with $n_1=\cdots=n_k=10$ observations per sample. We first generated the level 0.10 critical points for the test statistics considered here. For each of these settings we employed 10,000 replications in obtaining the level or power estimate. Since we took 0.10 as the nominal level of the tests, the estimated levels in Table 1 have standard deviation of $0.003=\{(0.10)(0.90)/10{,}000\}^{1/2}$. We then indicate, by $+$ $(-)$ signs, whenever the estimated level is more than two standard deviations above (below) 0.01.

4.2 *Discussion of the Results*

It is evident, upon examination of Table 1, that the statistics V_p^* and $V_{\max}^*$ do not hold their levels when the underlying distributions have different scale parameters, while the modified statistics $\hat{V}_p^*$ and $\hat{V}_{\max}^*$ maintain their levels well for all situations considered. The fact that the tests based on $\hat{V}_p^*$ and $\hat{V}_{\max}^*$ are exactly distribution-free when the populations are identical is also demonstrated in Table 1.

It can be seen from Table 2 that, for small sample sizes, the modified Hettmansperger-Norton tests exhibit slightly lower power for some situations than do the associated original tests. However, these power differences do not seem too costly a price to pay for the additional level holding properties of the modified tests.

The simulation results in Table 3 indicate that, for $1<p<k$, $\hat{V}_p^*$ is still superior to $\hat{A}_p^*$ for equal spacing umbrella alternatives. However, when the alternatives are not equally spaced, the test $\hat{V}_p^*$ may not be as powerful as $\hat{A}_p^*$. For the peak unknown setting, we observe that $\hat{V}_{\max}^*$ provides in general a better test than does $\hat{A}_{\max}^*$ for the cases with different scale parameters.

To conclude, we consider the modified Hettmansperger-Norton tests improvements over the corresponding original Hettmansperger-Norton tests because the modified tests are insensitive to differences in the scale parameters of the underlying symmetric distributions for holding their levels, and for small sample sizes the modified tests do not surrender significant amount of power relative to the associated unmodified tests. Moreover, when the umbrella patterned medians are believed to be equally spaced, we recommend use of the modified Hettmansperger-Norton tests instead of the modified Mack-Wolfe tests.

Table 3

Estimated levels for nominal $\alpha = 0.10$ when $k = 4$ amd $n_1 = n_2 = n_3 = n_4 = 10$

	σ_2/σ_1	σ_3/σ_1	σ_4/σ_1	$\hat{A}_3^*$	$\hat{A}_4^*$	$\hat{V}_3^*$	$\hat{V}_4^*$	$\hat{A}_{max}^*$	$\hat{V}_{max}^*$
	1	1	1	.100	.100	.099	.095	.100	.103
Normal	1	2	3	.101	.104	.099	.103	.100	.105
	2	1	3	.095	.108	.096	.102	.101	.104
Contaminated	1	1	1	.095	.100	.100	.102	.100	.105
Normal	1	2	3	.096	.104	.100	.105	.103	.107
	2	1	3	.093	.105	.093	.104	.104	.102
	1	1	1	.098	.102	.101	.103	.101	.104
Cauchy	1	2	3	.098	.105	.101	.103	.105	.103
	2	1	3	.091	.105	.094	.095	.105	.101

Table 4

Estimated powers for nominal $\alpha = 0.10$ when $k = 4$, $n_1 = n_2 = n_3 = n_4 = 10$

(a) Normal

σ_2/σ_1	σ_3/σ_1	σ_4/σ_1	$\theta_2-\theta_1$	$\theta_3-\theta_1$	$\theta_4-\theta_1$	$\hat{A}_p^*$	$\hat{V}_p^*$	$\hat{A}_{max}^*$	$\hat{V}_{max}^*$
1.0	1.0	1.0							
			0.0	0.0	1.0	.769	.777	.516	.546
			0.0	0.5	1.0	.866	.859	.662	.678
			0.5	1.0	1.5	.982	.984	.925	.939
			0.0	1.0	0.0	.868	.798	.703	.612
			0.0	1.0	0.5	.764	.801	.662	.654
			0.5	1.0	0.5	.762	.792	.633	.628
1.0	2.0	3.0							
			0.0	0.0	1.0	.321	.337	.151	.175
			0.0	0.5	1.0	.418	.418	.238	.276
			0.5	1.0	1.5	.630	.629	.484	.509
			0.0	1.0	0.0	.492	.491	.326	.330
			0.0	1.0	0.5	.428	.494	.332	.349
			0.5	1.0	0.5	.479	.529	.372	.382
2.0	1.0	3.0							
			0.0	0.0	1.0	.330	.336	.159	.174
			0.0	0.5	1.0	.470	.457	.301	.346
			0.5	1.0	1.5	.701	.687	.605	.703
			0.0	1.0	0.0	.769	.796	.592	.661
			0.0	1.0	0.5	.694	.797	.566	.619
			0.5	1.0	0.5	.694	.794	.552	.617

(b) Contaminated Normal

σ_2/σ_1	σ_3/σ_1	σ_4/σ_1	$\theta_2-\theta_1$	$\theta_3-\theta_1$	$\theta_4-\theta_1$	$\hat{A}_p^*$	$\hat{V}_p^*$	$\hat{A}_{max}^*$	$\hat{V}_{max}^*$
1.0	1.0	1.0							
			0.0	0.0	1.0	.669	.664	.402	.423
			0.0	0.5	1.0	.765	.769	.533	.555
			0.5	1.0	1.5	.938	.934	.808	.824
			0.0	1.0	0.0	.764	.691	.569	.492
			0.0	1.0	0.5	.655	.687	.535	.531
			0.5	1.0	0.5	.654	.695	.510	.525

b) Contaminated Normal (continuation)

σ_2/σ_1	σ_3/σ_1	σ_4/σ_1	$\theta_2-\theta_1$	$\theta_3-\theta_1$	$\theta_4-\theta_1$	$\hat{A}_p^*$	$\hat{V}_p^*$	$\hat{A}_{\max}^*$	$\hat{V}_{\max}^*$
1.0	2.0	3.0							
			0.0	0.0	1.0	.288	.292	.132	.157
			0.0	0.5	1.0	.374	.371	.203	.231
			0.5	1.0	1.5	.583	.576	.406	.427
			0.0	1.0	0.0	.420	.421	.277	.281
			0.0	1.0	0.5	.374	.418	.275	.294
			0.5	1.0	0.5	.411	.467	.314	.326
2.0	1.0	3.0							
			0.0	0.0	1.0	.294	.298	.137	.151
			0.0	0.5	1.0	.413	.399	.247	.282
			0.5	1.0	1.5	.610	.603	.487	.559
			0.0	1.0	0.0	.647	.667	.455	.462
			0.0	1.0	0.5	.574	.664	.436	.489
			0.5	1.0	0.5	.577	.661	.432	.490

(c) Cauchy

σ_2/σ_1	σ_3/σ_1	σ_4/σ_1	$\theta_2-\theta_1$	$\theta_3-\theta_1$	$\theta_4-\theta_1$	$\hat{A}_p^*$	$\hat{V}_p^*$	$\hat{A}_{\max}^*$	$\hat{V}_{\max}^*$
1.0	1.0	1.0							
			0.0	0.0	1.0	.426	.423	.207	.224
			0.0	0.5	1.0	.502	.492	.273	.290
			0.5	1.0	1.5	.693	.678	.454	.478
			0.0	1.0	0.0	.488	.435	.315	.277
			0.0	1.0	0.5	.407	.439	.300	.309
			0.5	1.0	0.5	.413	.436	.300	.295
1.0	2.0	3.0							
			0.0	0.0	1.0	.230	.228	.112	.124
			0.0	0.5	1.0	.286	.276	.152	.173
			0.5	1.0	1.5	.415	.422	.256	.283
			0.0	1.0	0.0	.288	.287	.190	.193
			0.0	1.0	0.5	.262	.289	.190	.201
			0.5	1.0	0.5	.284	.314	.214	.223
2.0	1.0	3.0							
			0.0	0.0	1.0	.233	.228	.111	.121
			0.0	0.5	1.0	.295	.288	.167	.186
			0.5	1.0	1.5	.431	.415	.285	.330
			0.0	1.0	0.0	.394	.407	.258	.265
			0.0	1.0	0.5	.351	.398	.245	.271
			0.5	1.0	0.5	.354	.408	.251	.278

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