HW#4 Survival Analysis I

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Problem 1.

The Clayton copula is defined as

$$C_\theta(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}, \qquad \theta>0.$$

(1) Make the contour plots of $C_\theta^{[1,1]}(u,v)\equiv\partial^2 C_\theta(u,v)/\partial u\partial v$ under $\theta=2$ and $\theta=8$.

Solution (1).

By straightforward calculations, we have

$$C_\theta^{[0,1]}(u,v)\equiv\partial C_\theta(u,v)/\partial v=v^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-1/\theta-1}$$

and

$$C_\theta^{[1,1]}(u,v)=\partial C_\theta^{[0,1]}(u,v)/\partial u=(\theta+1)(uv)^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-1/\theta-2}.$$

The contour plot of $C_\theta^{[1,1]}(u,v)$ is given in Figure 1.

(2) Make the scatter plots of (U_i,V_i), $i=1,2,\ldots,500$ under the Clayton copula with $\theta=2$ and $\theta=8$.

Solution (2).

To generate samples from the Clayton copula, we derive the conditional distribution of U given $V=v$. That is

$$\begin{aligned}\Pr(U\le u\mid V=v)&=\lim_{h\to 0}\frac{\Pr(U\le u,V\le v+h)-\Pr(U\le u,V\le v)}{\Pr(V\le v+h)-\Pr(V\le v)}\\&=\lim_{h\to 0}\frac{C_\theta(u,v+h)-C_\theta(u,v)}{h}=\frac{\partial}{\partial v}C_\theta(u,v)\\&=v^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-1/\theta-1}.\end{aligned}$$

Therefore, one can generate samples from the Clayton copula by using inverse transform method. We state the algorithm below.

Algorithm 1: Data generation algorithm

Step 1 Generate $V \sim U(0,1)$ and $W \sim U(0,1)$.

Step 2 Set $U = (V^{-\theta} W^{-\theta/(\theta+1)} - V^{-\theta} + 1)^{-1/\theta}$.

One can check the generated samples by examining the sample Kendall's tau. Under the Clayton copula, we have $\tau_\theta = \theta/(\theta+2)$. The sample Kendall's tau should be close to the theoretical value.

We repeat Algorithm 1 for 500 times to generate (U_i, V_i), $i = 1, 2, \ldots, 500$ under the Clayton copula. The scatter plots of (U_i, V_i), $i = 1, 2, \ldots, 500$ are given in Figure 2. Both Figure 1 and 2 show similar patterns. The plot under $\theta = 8$ $(\tau_\theta = 0.8)$ is more concentrate than the plot under $\theta = 2$ $(\tau_\theta = 0.5)$. The R codes are given in Appendix.

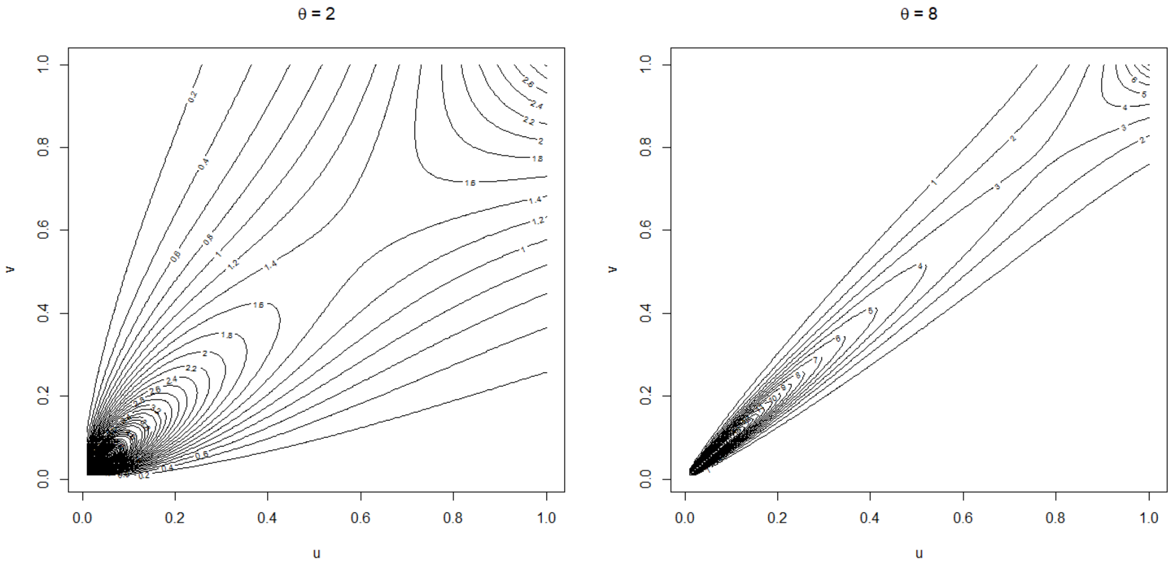

Figure 1. Contour plots of $C_\theta^{[1,1]}(u,v)$ under $\theta = 2$ and $\theta = 8$.

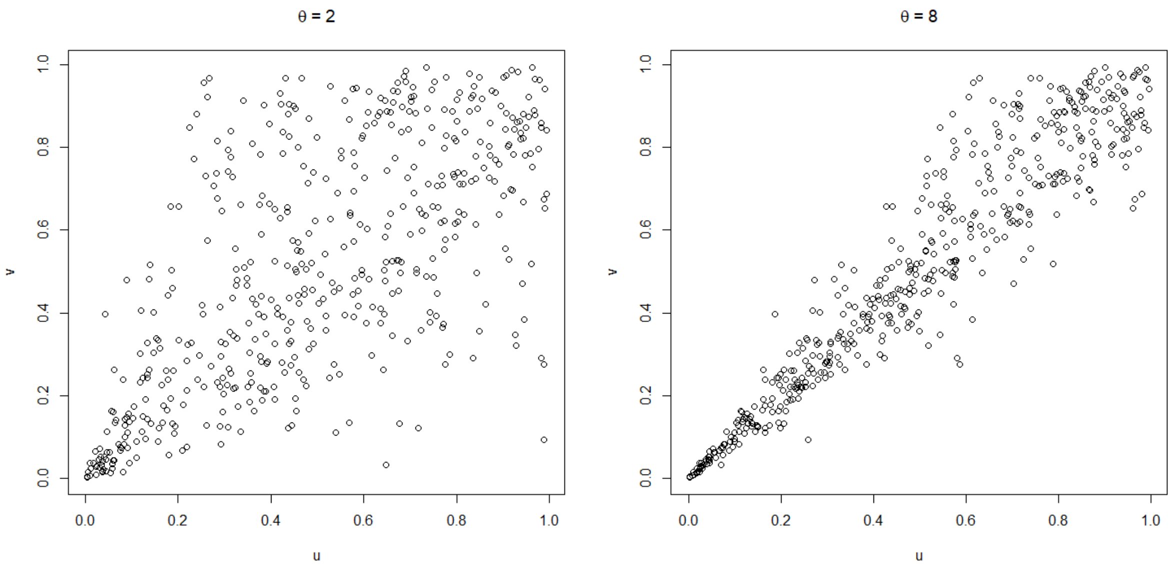

Figure 2. Scatter plots of (U_i, V_i), $i = 1, 2, \ldots, 500$ under $\theta = 2$ and $\theta = 8$.

Problem 2. [Exercise 3.6.1]

Show that Condition (C2') does not hold for $C_\theta(u,v)=(u^{-\theta}+v^{-\theta}-1)^{-1/\theta}$ with $-1<\theta<0$.

Solution 2.

From Problem 1, we know

$$C_\theta^{[1,1]}(u,v)=(\theta+1)(uv)^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-1/\theta-2}.$$

Since $0\le u\le 1$, $0\le v\le 1$, and $-1<\theta<0$, we have

$$(\theta+1)(uv)^{-\theta-1}(u^{-\theta}+v^{-\theta}-1)^{-1/\theta-2}\ge 0 \Leftrightarrow (u^{-\theta}+v^{-\theta}-1)^{-1/\theta-2}\ge 0.$$

To disprove (C2'), it suffices to find a counter example which does not satisfy the last inequality. One may choose $u=0.1$, $v=0.1$, and $\theta=-1/3$, then

$$(0.1^{1/3}+0.1^{1/3}-1)^{3-2}=-0.0717<0.$$

This completes the proof.

Problem 3. [Exercise 3.6.4]

Show that the copula density for an Archimedean copula is expressed as

$$C_\theta^{[1,1]}(u,v)=-\frac{\phi_\theta''\{C_\theta(u,v)\}\phi_\theta'(u)\phi_\theta'(v)}{[\phi_\theta'\{C_\theta(u,v)\}]^3}.$$

Solution 3.

The Archimedean copula is expressed as

$$C_\theta(u,v)=\phi_\theta^{-1}\{\phi_\theta(u)+\phi_\theta(v)\},\qquad 0\le u\le 1,\ 0\le v\le 1,$$

where θ is a parameter and ϕ_θ is a generator. This implies

$$\phi_\theta\{C_\theta(u,v)\}=\phi_\theta(u)+\phi_\theta(v).$$

Suppose that the generator function is twice differentiable, we have

$$\phi_\theta'\{C_\theta(u,v)\}\frac{\partial}{\partial u}C_\theta(u,v)=\phi_\theta'(u),\qquad \phi_\theta'\{C_\theta(u,v)\}\frac{\partial}{\partial v}C_\theta(u,v)=\phi_\theta'(v),$$

and

$$\phi''_\theta\{C_\theta(u,v)\}\frac{\partial}{\partial v}C_\theta(u,v)\frac{\partial}{\partial u}C_\theta(u,v)+\phi'_\theta\{C_\theta(u,v)\}\frac{\partial^2}{\partial u\partial v}C_\theta(u,v)=0\,.$$

The last equality can be written as

$$\phi'_\theta\{C_\theta(u,v)\}\frac{\partial^2}{\partial u\partial v}C_\theta(u,v)=-\frac{\phi''_\theta\{C_\theta(u,v)\}}{\phi'_\theta\{C_\theta(u,v)\}}\frac{\partial}{\partial v}C_\theta(u,v)\frac{\partial}{\partial u}C_\theta(u,v)\,.$$

Applying

$$\frac{\partial}{\partial u}C_\theta(u,v)=\frac{\phi'_\theta(u)}{\phi'_\theta\{C_\theta(u,v)\}}\quad\text{and}\quad\frac{\partial}{\partial v}C_\theta(u,v)=\frac{\phi'_\theta(v)}{\phi'_\theta\{C_\theta(u,v)\}}\,,$$

one can show that

$$C_\theta^{[1,1]}(u,v)=\frac{\partial^2}{\partial u\partial v}C_\theta(u,v)=-\frac{\phi''_\theta\{C_\theta(u,v)\}\phi'_\theta(u)\phi'_\theta(v)}{[\,\phi'_\theta\{C_\theta(u,v)\}\,]^3}\,.$$

Then we have finished the proof. Another observation in this exercise is the relationship

$$\phi'_\theta(u)\frac{\partial}{\partial v}C_\theta(u,v)=\phi'_\theta(v)\frac{\partial}{\partial u}C_\theta(u,v)\,.$$

Problem 4. [Exercise 3.6.5]

Calculate Kendall's tau for the Clayton, Gumbel, FGM, and Joe copulas.

Solution 4.

(I) The Clayton copula:

Its generator is $\phi_\theta(t)=(t^{-\theta}-1)/\theta$ with first derivative $\phi'_\theta(t)=-t^{-\theta-1}$. Then, Kendall's tau under the Clayton copula is

$$\begin{aligned}1-4\int_0^1\frac{\phi_\theta(t)}{\phi'_\theta(t)}dt&=1+4\int_0^1\frac{(t^{-\theta}-1)/\theta}{-t^{-\theta-1}}dt=1-\frac{4}{\theta}\int_0^1(t-t^{\theta+1})dt\\&=1-\frac{4}{\theta}\left(\frac{1}{2}t^2-\frac{1}{\theta+2}t^{\theta+2}\right)\Bigg|_0^1=1-\frac{4}{\theta}\left(\frac{1}{2}-\frac{1}{\theta+2}\right)\\&=\frac{\theta}{\theta+2}.\end{aligned}$$

(II) The Gumbel copula:

Its generator is $\phi_\theta(t)=\{-\log(t)\}^{\theta+1}$ with first derivative $\phi'_\theta(t)=-(\theta+1)\{-\log(t)\}^\theta/t$.

Then, Kendall's tau under the Gumbel copula is

$$1-4\int_0^1\frac{\phi_\theta(t)}{\phi'_\theta(t)}dt=1+4\int_0^1\frac{\{-\log(t)\}^{\theta+1}}{-(\theta+1)\{-\log(t)\}^\theta/t}dt=1+\frac{4}{\theta+1}\int_0^1 t\log(t)dt.$$

Consider change of variables $x=\log(t)$ and $dt=e^x dx$. Then,

$$\int_0^1 t\log(t)dt=\int_{-\infty}^0 xe^{2x}dx=\int_\infty^0 xe^{-2x}dx=-\int_0^\infty xe^{-2x}dx=-\frac{1}{2}\int_0^\infty 2xe^{-2x}dx=-\frac{1}{4}.$$

We obtain

$$1+\frac{4}{\theta+1}\left(-\frac{1}{4}\right)=1-\frac{1}{\theta+1}=\frac{\theta}{\theta+1}.$$

(III) The FGM copula:

The FGM copula is defined as

$$C_\theta(u,v)=uv\{1+\theta(1-u)(1-v)\},\qquad -1\le\theta\le 1.$$

It copula density is

$$C_\theta^{[1,1]}(u,v)=\frac{\partial^2}{\partial u\partial v}C_\theta(u,v)=1+\theta(1-2u)(1-2v).$$

By straightforward calculations, Kendall's tau under the FGM copula is

$$\begin{aligned}
&4\int_0^1\int_0^1 C_\theta(u,v)C_\theta^{[1,1]}(u,v)dudv-1\\
&=4\int_0^1\int_0^1\{uv+\theta u(1-u)v(1-v)\}\{1+\theta(1-2u)(1-2v)\}dudv-1\\
&=4\int_0^1\int_0^1\{uv+\theta u(1-2u)v(1-2v)+\theta u(1-u)v(1-v)\\
&\qquad\qquad+\theta^2u(1-u)(1-2u)(1-2v)v(1-v)\}dudv-1.
\end{aligned}$$

We calculate the integral separately. We have

$$\int_0^1 udu\int_0^1 vdv=\frac{1}{4}, \quad \int_0^1 u(1-2u)du\int_0^1 v(1-2v)dv=\frac{1}{36}, \quad \int_0^1 u(1-u)du\int_0^1 v(1-v)dv=\frac{1}{36},$$

and

$$\int_0^1 u(1-u)(1-2u)du\int_0^1 v(1-v)(1-2v)dv=0.$$

Finally,

$$4\int_0^1\int_0^1 C_\theta(u,v)C_\theta^{[1,1]}(u,v)dudv-1=1+\frac{\theta}{9}+\frac{\theta}{9}+0-1=\frac{2\theta}{9}.$$

(IV) The Joe copula:

Its generator is $\phi_\theta(t)=-\log\{1-(1-t)^\theta\}$ with inverse function $\phi_\theta^{-1}(s)=1-(1-e^{-s})^{1/\theta}$.

Its first derivative is

$$\frac{d}{ds}\phi_\theta^{-1}(s)=\frac{d}{ds}1-(1-e^{-s})^{1/\theta}=\frac{1}{\theta}(1-e^{-s})^{1/\theta-1}e^{-s}.$$

Then, Kendall’s tau under the Joe copula is

$$1-4\int_0^\infty s\left\{\frac{d}{ds}\phi_\theta^{-1}(s)\right\}^2 ds=1-4\int_0^\infty \frac{s(1-e^{-s})^{2/\theta-2}e^{-2s}}{\theta^2}ds.$$

This is an alternative formula for Kendall’s tau under the Archimedean copula.

Appendix

```
c.Clayton = function(u,v) {

  return((theta+1)*(u*v)^(-theta-1)*(u^(-theta)+v^(-theta)-1)^(-1/theta-2))

}

u.v = v.v = seq(0.01,1,length.out = 200)
M = matrix(0,length(u.v),length(v.v))

theta = 2
i = 1
for (u in u.v) {

  j = 1
  for (v in v.v) {

    M[i,j] = c.Clayton(u,v)
    j = j+1

  }
  i = i+1

}

n = 500
set.seed(816)
v = runif(n); w = runif(n)
u = (v^(-theta)*w^(-theta/(theta+1))-v^(-theta)+1)^(-1/theta)

cor(u,v,method = "kendall"); theta/(theta+2)

contour(u.v,v.v,M,nlevels = 200,xlab = expression(u),ylab = expression(v),
        main = expression(theta*" = 2"))
plot(u,v,xlab = expression(u),ylab = expression(v),
     main = expression(theta*" = 2"))
```