HOMEWORK#1

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Exercise 2.1

The lifetime of light bulbs follows an exponential distribution with a hazard rate of 0.001 failures per hour of use

(a) Find the mean lifetime of a randomly selected light bulb.

Answer:

Let X denote the lifetime of light bulbs,then the hazard rate h(x) = 0.001.

So the survival function S(x) = $\exp\{-\int_0^x \frac{1}{1000} du\} = \exp\{\frac{-x}{1000}\}$, and the p.d.f. of X

is f(x) = $-\frac{d}{dx}S(x) = \frac{1}{1000}\cdot\exp\{\frac{-x}{1000}\}$; x > 0 .

$\rightarrow$ X has exponential distribution with $\lambda = \frac{1}{1000} \rightarrow \therefore E(x) = \frac{1}{\lambda} = 1000$.

(b) Find the median lifetime of a randomly selected light bulb.

Answer:

Set $S(m) = \exp\{\frac{-m}{1000}\} = 0.5 \rightarrow \frac{-m}{1000} = \log(0.5) \rightarrow m = -1000\cdot\log(0.5) = 301.03$

$\therefore$ median lifetime is 301.03 hours

(c) What is the probability a light bulb will still function after 2,000 hours of use?

Answer:

$P\{X>2000\} = S(x=2000) = \exp\{\frac{-2000}{1000}\} = e^{-2} = 0.135$

Exercise 2.2

The time in days to development of a tumor for rats exposed to a carcinogen follows a Weibull distribution with α=2 and λ=0.001.

(a) What is the probability a rat will be tumor free at 30 days? 45 days? 60 days?

Answer:

Let X denote the time in days to development of a tumor for rats exposed to a carcinogen, and X has Weibull distribution with α =2 and λ =0.001.

So the p.d.f. of X is f(x) = $\frac{2}{1000}\cdot x\cdot\exp\{\frac{-1}{1000}\cdot x^2\}$.

$P\{X>t\} = \int_t^\infty \frac{2}{1000}\cdot x\cdot\exp\{\frac{-1}{1000}\cdot x^2\}\,dx = -\exp\{\frac{-1}{1000}\cdot x^2\}|_t^\infty = \exp\{\frac{-1}{1000}\cdot t^2\}$

$\therefore$ $p\{X>30\} = \exp\{\frac{-1}{1000}\cdot 30^2\} = 0.407$; $p\{X>45\} = \exp\{\frac{-1}{1000}\cdot 45^2\} = 0.132$

$p\{X>60\} = \exp\{\frac{-1}{1000}\cdot 60^2\} = 0.027$

(b) What is the mean time to tumor?

Answer:

Because X has Weibull distribution with α =2 and λ =0.001.

∴ E(X) = $(\frac{1}{1000})^{-\frac{1}{2}} \cdot \Gamma\,(1+\frac{1}{2})$ = $\sqrt{1000} \cdot \Gamma\,(\frac{1}{2}) \cdot \frac{1}{2}$ = 28.025

(b) Find the hazard rate of the time to tumor appearance at 30 days, 45 days, and 60 days.

Answer:

Because S(x) = P{X>x} = $\exp\{\frac{-1}{1000} \cdot x^2\}$, hazard rate is h(x) = - $\frac{d}{dx}$logS(x).

→ h(x) = $\frac{1}{500}x$ ∴ h(30) = $\frac{30}{500}$ =0.06 ; h(45) = $\frac{45}{500}$ =0.09 ; h(30) = $\frac{60}{500}$ =0.12

(d) Find the median time to tumor.

Answer:

Set S(m) =$\exp\{\frac{-1}{1000} \cdot m^2\}$ = 0.5 →m^2= -1000·log(0.5) →m=$\sqrt{-1000 \cdot log(0.5)}$

∴ tne median time to tumor is 26.32769 days

Exercise 2.3

The time to death (in days) following a kidney transplant follows a log logistic distribution with α=1.5 and λ= 0.01.

(a) Find the 50, 100, and 150 day survival probabilities for kidney transplantation in patients.

Answer:

Let X denote the time to death following a kidney transplant,because X has log logistic distribution with α=1.5 and λ= 0.01,the p.d.f of X is f(x) = $\frac{0.015 \cdot x^{0.5}}{(1+0.01 \cdot x^{1.5})^2}$

The survival function S(x) = P{X>x} = $\int_x^{\infty} \frac{0.015 \cdot u^{0.5}}{(1+0.01 \cdot u^{1.5})^2} du$ = $\frac{-1}{1+0.01 \cdot u^{1.5}} |_x^{\infty}$

= $\frac{1}{1+0.01 \cdot x^{1.5}}$ ∴ P{X>50} = S(50) = $\frac{1}{1+0.01 \cdot 50^{1.5}}$ = 0.220

P{X>100} = $\frac{1}{1+0.01 \cdot 100^{1.5}}$ = 0.091 ; P{X>150} = $\frac{1}{1+0.01 \cdot 150^{1.5}}$ = 0.052

(b) Find the median time to death following a kidney transplant.

Answer:

Set S(m) = $\frac{1}{1+0.01 \cdot m^{1.5}}$ = 0.5 → $1 + 0.01 \cdot m^{1.5}$ = 2 → m = $\sqrt[1.5]{100}$ = 21.544

∴ the median time to death is 21.544 days

(c) Show that the hazard rate is initially increasing and, then, decreasing over time. Find the time at which the hazard rate changes from increasing to decreasing.

Answer:

Hazard rate h(x)= $\frac{f(x)}{S(x)} = \frac{0.015\cdot x^{0.5}}{(1+0.01\cdot x^{1.5})^2} \times (1+0.01\cdot x^{1.5}) = \frac{0.015\cdot x^{0.5}}{1+0.01\cdot x^{1.5}}$

h'(x) = $\frac{d}{dx}\frac{0.015\cdot x^{0.5}}{1+0.01\cdot x^{1.5}} = \frac{0.015\cdot x^{-0.5}\cdot(0.5-0.01\cdot x^{1.5})}{(1+0.01\cdot x^{1.5})^2}$ to solve h'(x) = 0

$\rightarrow \frac{0.015\cdot x^{-0.5}\cdot(0.5-0.01\cdot x^{1.5})}{(1+0.01\cdot x^{1.5})^2} = 0 \rightarrow 0.5-0.01\cdot x^{1.5} = 0 \rightarrow x = \sqrt[1.5]{50} = 13.572$

We can know if x > 13.572,then h'(x) will be negative;similarly,if x < 13.572,then h'(x) will be positive.

∴ h(x) is increasing if 0 < x < 13.572 ; h(x) is decreasing if x > 13.572

And the change point is x = 13.572

(d) Find the mean time to death.

Answer:

f(x) = $\frac{0.015\cdot x^{0.5}}{(1+0.01\cdot x^{1.5})^2} \rightarrow$ E(x) = $\int_0^\infty x\cdot\frac{0.015\cdot x^{0.5}}{(1+0.01\cdot x^{1.5})^2}\,dx$

let u = $\frac{1}{1+0.01\cdot x^{1.5}}$ du = $\frac{-0.015\cdot x^{0.5}}{(1+0.01\cdot x^{1.5})^2}$ dx x = $\sqrt[1.5]{100\cdot(\frac{1}{u}-1)}$ 0<u<1

E(x) = $\int_0^1 \sqrt[1.5]{100\cdot(\frac{1}{u}-1)}\,du = 21.544\cdot\int_0^1(\frac{1}{u}-1)^{\frac{1}{1.5}}\,du$

= $21.544\cdot\int_0^1 u^{\frac{-1}{1.5}}(1-u)^{\frac{1}{1.5}}\,du$ by beta integral

E(x) = $21.544\cdot\int_0^1 u^{\frac{0.5}{1.5}-1}(1-u)^{\frac{2.5}{1.5}-1}\,du = 21.544\cdot\frac{\Gamma(\frac{0.5}{1.5})\Gamma(\frac{2.5}{1.5})}{\Gamma(\frac{0.5}{1.5}+\frac{2.5}{1.5})}$ = 21.544 · 2.418

∴ the mean time to death is 52.102 days

Exercise 2.4

A model for lifetimes, with a bathtub-shaped hazard rate, is the exponential power distribution with survival function S(x)=exp{$1-\exp[(\lambda x)^\alpha]$}

(a) If α= 0.5, show that the hazard rate has a bathtub shape and find the time at which the hazard rate changes from decreasing to increasing.

Answer:

h(x) = $-\frac{d}{dx}$logS(x) = $-\frac{d}{dx}$(1- $\exp[(\lambda x)^{\frac{1}{2}}]$) = $\frac{1}{2}\cdot(\frac{\lambda}{x})^{\frac{1}{2}}\cdot\exp[(\lambda x)^{\frac{1}{2}}]$

h'(x) = $\frac{d}{dx}\frac{1}{2}\cdot(\frac{\lambda}{x})^{\frac{1}{2}}\cdot\exp[(\lambda x)^{\frac{1}{2}}] = \frac{-1}{4}\cdot\lambda^{\frac{1}{2}}\cdot x^{\frac{-3}{2}}\cdot\exp\left[(\lambda x)^{\frac{1}{2}}\right]+\frac{1}{4}\cdot\frac{\lambda}{x}\cdot\exp[(\lambda x)^{\frac{1}{2}}]$

let h'(x) = 0 $\rightarrow$ $\frac{1}{4} \cdot \frac{\lambda}{x} \cdot \exp\left[(\lambda x)^{\frac{1}{2}}\right] = \frac{1}{4} \cdot \lambda^{\frac{1}{2}} \cdot x^{\frac{-3}{2}} \cdot \exp\left[(\lambda x)^{\frac{1}{2}}\right]$ $\rightarrow$ x = $\frac{1}{\lambda}$

if x > $\frac{1}{\lambda}$,then h'(x) > 0 $\rightarrow$ h(x) is increasing ; if x < $\frac{1}{\lambda}$,then h'(x) < 0 $\rightarrow$ h(x) is decreasing ∴ the change point is x = $\frac{1}{\lambda}$

(b) If α= 2, show that the hazard rate of x is monotone increasing.

Answer:

h(x) = - $\frac{d}{dx}$logS(x) = - $\frac{d}{dx}$(1- $\exp[(\lambda x)^2]$) = $2 \cdot \lambda^2 \cdot x \cdot \exp[(\lambda x)^2]$

h'(x) = $\frac{d}{dx} 2 \cdot \lambda^2 \cdot x \cdot \exp[(\lambda x)^2] = 2 \cdot \lambda^2 \cdot \exp[(\lambda x)^2] + 4 \cdot \lambda^4 \cdot x^2 \cdot \exp[(\lambda x)^2]$

because $2 \cdot \lambda^2 \cdot \exp[(\lambda x)^2]$ > 0 and $4 \cdot \lambda^4 \cdot x^2 \cdot \exp[(\lambda x)^2]$ > 0 for all x and λ ,
we can obtain that h'(x) > 0 fot all x and λ if α =2 $\rightarrow$ h(x) is monotone increasing
∴ the hazard rate of x is monotone increasing.

Exercise 2.5

The time to death (in days) after an autologous bone marrow transplant, follows a log normal distribution with μ= 3.177 and σ = 2.084. Find

(a) the mean and median times to death;

Answer:

Let X denote the time to death after an autologous bone marrow transplant,because, X has log normal distribution, and
let X = e^Y where Y~N(μ =3.177, σ =2.084) the mean is E(X)

E(X) = E(e^Y) = $\exp\{\mu + \frac{1}{2} \cdot \sigma^2\}$ = $\exp\{3.177 + \frac{1}{2} \cdot 2.084^2\}$ = 210.299

S(x) = P{X>x} = P{$e^Y > x$} = P{Y>logx} = P{$\frac{Y-3.177}{2.084} > \frac{logx-3.177}{2.084}$} = P{Z > $\frac{logx-3.177}{2.084}$}

Where Z~N(0,1), now denote Φ(t) = P{Z<t} ,so S(x) = 1-$\Phi(\frac{logx-3.177}{2.084})$

Let S(m) = 1-$\Phi(\frac{logm-3.177}{2.084})$ = $\frac{1}{2}$ $\rightarrow \Phi(\frac{logm-3.177}{2.084})$ = $\frac{1}{2}$ $\rightarrow$ $\frac{logm-3.177}{2.084} = \Phi^{-1}(\frac{1}{2}) = 0$

→m = exp{3.177} = 23.975 ∴ the median time is 23.975 days

(b) the probability a individual survives 100, 200, and 300 days following a transplant

Answer:

P{X>100}=S(100)=1-$\Phi(\frac{log100-3.177}{2.084})$=0.247(by check the normal distribution table)

P{X>200}=1-$\Phi(\frac{log200-3.177}{2.084})$=0.154 P{X>300}=1-$\Phi(\frac{log300-3.177}{2.084})$=0.113

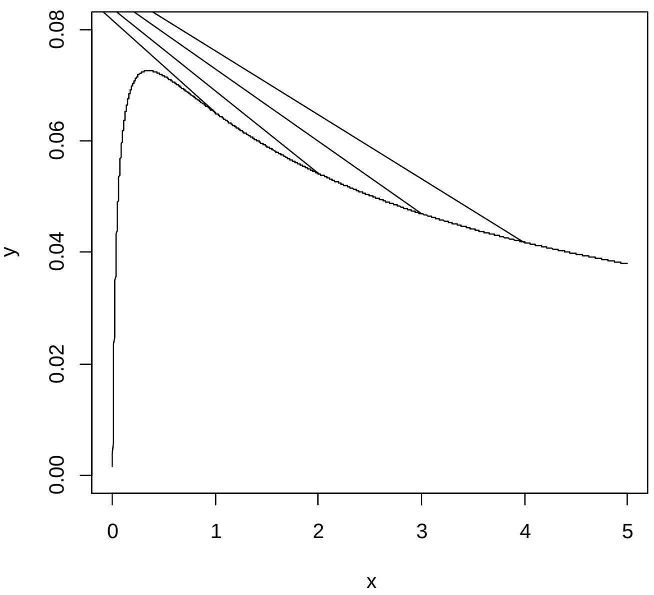
(c) plot the hazard rate of the time to death and interpret the shape of this function.

Answer:

$$h(x) = -\frac{d}{dx}\log S(x) = -\frac{d}{dx}\log(1-\Phi(\frac{\log x-3.177}{2.084})) = \frac{\frac{1}{2.048x}\cdot\phi(\frac{\log x-3.177}{2.084})}{1-\Phi(\frac{\log x-3.177}{2.084})}$$

where $\phi(.)$ is p.d.f. of Z

graph:



R-code:

```
x=seq(0,5,0.001);
y=exp(-((log(x)-3.177)/2.084)^2/2)/(sqrt(2*pi)*2.048*x*(1-pnorm((log(x)-3.17
7)/2.084,0,1)));
plot(x,y,type="l",xlim=c(0,5),ylim=c(0,0.08))
```

h(x) is increasing,when x >0.3470379 ,h(x) is decreasing.

Exercise 2.8

The battery life of an internal pacemaker, in years, follows a Pareto distribution with θ= 4 and λ= 5.

(a) What is the probability the battery will survive for at least 10 years?

Answer:

Let X denote the battery life,because X has Pareto distribution,the p.d.f of X is given by f(x) = $4 \cdot 5^4 \cdot x^{-5}$; x > 5

P{X > 10} = $\int_{10}^{\infty} 4 \cdot 5^4 \cdot x^{-5}\,dx = -5^4 \cdot x^{-4}|_{10}^{\infty}$ = 0.0625

(b) What is the mean time to battery failure?

Answer:

$$E(X) = \int_5^\infty x \cdot 4 \cdot 5^4 \cdot x^{-5}\, dx = \int_5^\infty 4 \cdot 5^4 \cdot x^{-4}\, dx = -4 \cdot 5^4 \cdot \frac{1}{3} \cdot x^{-3}|_5^\infty = 6.667$$

∴ the mean time to battery failure is 6.667 years.

(c) If the battery is scheduled to be replaced at the time t_0, at which 99% of all batteries have yet to fail (that is, at t_0 so that Pr $(X > t_0)$ =0 .99), find t_0.

Answer:

$$P\{X > t_0\} = \int_{t0}^\infty 4 \cdot 5^4 \cdot x^{-5}\, dx = -5^4 \cdot x^{-4}|_{t0}^\infty = 5^4 \cdot t_0^{-4} = 0.99$$

$$\rightarrow 5^4 \cdot \frac{100}{99} = t_0^4 \quad \rightarrow \quad t_0 = \sqrt[4]{5^4 \cdot \frac{100}{99}} = 5.012579 \text{ (years)}$$

Exercise 2.10

A model used in the construction of life tables is a piecewise, constant hazard rate model. Here the time axis is divided into k intervals, $[\tau_{i-1}, \tau_i)$, i = 1, . . . , k, with $\tau_0 = 0$ and $\tau_k = \infty$. The hazard rate on the ith interval is a constant value, θ_i; that is

$$h(x) = \begin{cases} \theta_1 & 0 \le x < \tau_1 \\ \theta_2 & \tau_1 \le x < \tau_2 \\ \vdots & \\ \theta_{k-1} & \tau_{k-2} \le x < \tau_{k-1} \\ \theta_k & \tau_{k-1} \le x \end{cases}$$

(a) Find the survival function for this model.

Answer:

For $0 \le x < \tau_1$ $S(x) = \exp\{-\int_0^x h(u)\, du\} = \exp\{-\int_0^x h(u)\, du\} = \exp\{-\theta_1 \cdot x\}$

For $\tau_1 \le x < \tau_2$ $S(x) = \exp\{-\int_{\tau_1}^x h(u)\, du\} \cdot \exp\{-\int_0^{\tau_1} h(u)\, du\}$

$$= \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x - \tau_1)]\}$$

For $\tau_{k-2} \le x < \tau_{k-1}$

$$S(x) = \exp\{-\int_{\tau_{k-2}}^x h(u)\, du\} \cdot \exp\{-\int_{\tau_{k-3}}^{\tau_{k-2}} h(u)\, du\} \times \cdots \times \exp\{-\int_0^{\tau_1} h(u)\, du\}$$

$$= \exp\{-\theta_{k-1} \cdot (x - \tau_{k-2})\} \cdot \exp\{-\theta_{k-2} \cdot (\tau_{k-2} - \tau_{k-3})\} \times \cdots \times \exp\{-\theta_1 \cdot \tau_1\}$$

$$\text{So } S(x) = \begin{cases} \exp\{-\theta_1 \cdot x\} & \text{For } 0 \le x < \tau_1 \\ \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x - \tau_1)]\} & \text{For } \tau_1 \le x < \tau_2 \\ \vdots & \vdots \\ \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x - \tau_{k-2})]\} & \text{For } \tau_{k-2} \le x < \tau_{k-1} \\ \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_k \cdot (x - \tau_{k-1})]\} & \text{For } \tau_{k-1} \le x \end{cases}$$

(b) Find the mean residual-life function.

Answer:

Mean residual-life function $e(x) = \dfrac{\int_x^\infty S(t)\,dt}{S(x)}$

For $0 \le x < \tau_1$ $\quad e(x) = \dfrac{\int_x^\infty \exp\{-\theta_1 \cdot t\}dt}{\exp\{-\theta_1 \cdot x\}} = \dfrac{\int_x^{\tau_1} \exp\{-\theta_1 \cdot t\}dt + \int_{\tau_1}^\infty S(t)\,dt}{\exp\{-\theta_1 \cdot x\}}$

$$= \frac{\exp\{-\theta_1 \cdot t\}}{\exp\{-\theta_1 \cdot x\}} \cdot \frac{-1}{\theta_1} \Big|_x^{\tau_1} + \frac{\int_{\tau_1}^\infty S(t)\,dt}{\exp\{-\theta_1 \cdot x\}}$$

$$= \frac{-\exp\{-\tau_1 \cdot \theta_1\} + \exp\{-x \cdot \theta_1\}}{\theta_1 \cdot \exp\{-\theta_1 \cdot x\}} + \frac{\int_{\tau_1}^\infty S(t)\,dt}{\exp\{-\theta_1 \cdot x\}}$$

For $\tau_1 \le x < \tau_2$

$$e(x) = \frac{\int_x^\infty \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(t-\tau_1)]\}\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}} = \frac{\int_x^{\tau_2} \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(t-\tau_1)]\}dt + \int_{\tau_2}^\infty S(t)\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}}$$

$$= \frac{\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(t-\tau_1)]\}}{\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}} \cdot \frac{-1}{\theta_2} \Big|_x^{\tau_2} + \frac{\int_{\tau_2}^\infty S(t)\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}}$$

$$= \frac{-\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(\tau_2-\tau_1)]\} + \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\} + \theta_2 \int_{\tau_2}^\infty S(t)\,dt}{\theta_2 \cdot \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}}$$

For $\tau_{k-2} \le x < \tau_{k-1}$

$$e(x) = \frac{\int_x^\infty \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (t-\tau_{k-2})]\}\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}}$$

$$= \frac{\int_x^{\tau_{k-1}} \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (t-\tau_{k-2})]\}\,dt + \int_{\tau_{k-1}}^\infty S(t)\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}}$$

$$= \frac{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (t-\tau_{k-2})]\}}{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}} \cdot \frac{-1}{\theta_{k-1}} \Big|_x^{\tau_{k-1}} + \frac{\int_{\tau_{k-1}}^\infty S(t)\,dt}{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}}$$

$$= \frac{-\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (\tau_{k-1}-\tau_{k-2})]\} + \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\} + \theta_{k-1} \int_{\tau_{k-1}}^\infty S(t)\,dt}{\theta_{k-1} \cdot \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}}$$

$\therefore e(x) =$

$$\begin{cases} \dfrac{-\exp\{-\tau_1 \cdot \theta_1\} + \exp\{-x \cdot \theta_1\}}{\theta_1 \cdot \exp\{-\theta_1 \cdot x\}} + \dfrac{\int_{\tau_1}^\infty S(t)\,dt}{\exp\{-\theta_1 \cdot x\}} & \text{For } 0 \le x < \tau_1 \\ \dfrac{-\exp\{-[\theta_1 \cdot \tau_1 + \theta_2(\tau_2-\tau_1)]\} + \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\} + \theta_2 \int_{\tau_2}^\infty S(t)\,dt}{\theta_2 \cdot \exp\{-[\theta_1 \cdot \tau_1 + \theta_2(x-\tau_1)]\}} & \text{For } \tau_1 \le x < \tau_2 \\ \vdots & \\ \dfrac{-\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (\tau_{k-1}-\tau_{k-2})]\} + \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\} + \theta_{k-1} \int_{\tau_{k-1}}^\infty S(t)\,dt}{\theta_{k-1} \cdot \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (x-\tau_{k-2})]\}} & \text{For } \tau_{k-2} \le x < \tau_{k-1} \\ \dfrac{\exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_k \cdot (x-\tau_{k-1})]\}}{\theta_k \cdot \exp\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_k \cdot (x-\tau_{k-1})]\}} = \dfrac{1}{\theta_k} & \text{For } \tau_{k-1} \le x \end{cases}$$

(c) Find the median residual-life function.

Answer:

Set S(m) =0.5,if $\tau_{i-2} \leq m < \tau_{i-1}$;1<i<k

→exp$\{-[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-1} \cdot (m - \tau_{k-2})]\}$=0.5

→exp$\{-[\theta_{k-1} \cdot (m - \tau_{k-2})]\}$ = exp$\{[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-2} \cdot (\tau_{k-2} - \tau_{k-3})]\}$ ·0.5

∴ m = $-\frac{1}{\theta_{k-1}}\{[\theta_1 \cdot \tau_1 + \cdots + \theta_{k-2} \cdot (\tau_{k-2} - \tau_{k-3})] + \log(0.5)\}$+$\tau_{k-2}$

Exercise 2.11

In some applications, a third parameter, called a guarantee time, is included in the models discussed in this chapter. This parameter ϕ is the smallest time at which a failure could occur. The survival function of the three-parameter Weibull distribution is given by S(x) =$\begin{cases} 1 & if\ x < \phi \\ \exp\{-\lambda(x-\phi)^\alpha\} & if\ x \geq \phi \end{cases}$

(a) Find the hazard rate and the density function of the three- parameter Weibull distribution

Answer:

h(x) =$-\frac{d}{dx} logS(x)$ = $\begin{cases} 0 & if\ x < \phi \\ \alpha\lambda(x-\phi)^{\alpha-1} & if\ x \geq \phi \end{cases}$

because h(x) = $\frac{f(x)}{S(x)}$ → f(x) = h(x)·S(x)

so f(x) = $\begin{cases} 0 & if\ x < \phi \\ \alpha\lambda(x-\phi)^{\alpha-1} \exp\{-\lambda(x-\phi)^\alpha\} & if\ x \geq \phi \end{cases}$

(b) Suppose that the survival time X follows a three-parameter Weibull distribution with α = 1, λ = 0.0075 and ϕ = 100. Find the mean and median lifetimes

Answer:

E(X) = $\int_\phi^\infty x \cdot \alpha\lambda(x-\phi)^{\alpha-1} \exp\{-\lambda(x-\phi)^\alpha\}\, dx$ let u = x- ϕ 0<u

E(X) = $\int_0^\infty (u+\phi) \cdot \alpha\lambda u^{\alpha-1} \exp\{-\lambda u^\alpha\}\, du$

= $\int_0^\infty \alpha\lambda u^\alpha \exp\{-\lambda u^\alpha\}\, du + \int_0^\infty \phi \cdot \alpha\lambda u^{\alpha-1} \exp\{-\lambda u^\alpha\}\, du$

(by gamma integral)

= $\lambda^{\frac{-1}{\alpha}} \cdot \Gamma\left(1+\frac{1}{\alpha}\right) + \phi$ = $0.0075^{-1} \cdot \Gamma(1+1) + 100 = 233.334$

Set S(m) =exp$\{-0.0075(m-100)^1\}$ = 0.5 → m = log(0.5)·$\frac{-1}{0.0075} + 100$

→ m = 192.420 ∴ the median lifetimes is 192.420