Quiz #2, Survival Analysis I, 2016 Spring

Name: 蔡佩洹 Tsai Pei-Yuan 103225011

5/5

- **Not only answer but also derivations**

+3/3

1. Let (T_1, T_2) be a bivariate lifetime with

$$\Pr(T_1 \ge t_1, T_2 \ge t_2) = \left[\exp\left\{\theta\left(\frac{t_1}{\alpha_1}\right)^{\beta_1}\right\} + \exp\left\{\theta\left(\frac{t_2}{\alpha_2}\right)^{\beta_2}\right\} - 1\right]^{-\frac{1}{\theta}}, \qquad t_1, t_2 \ge 0.$$

Let $T = \min(T_1, T_2)$ be a lifetime and $C = \begin{cases} 1 & \text{if} \quad T_1 \le T_2 \\ 2 & \text{if} \quad T_1 > T_2 \end{cases}$ be the failure mode.

+1 (i) Calculate the marginal survival function $S_1(t_1) = \Pr(T_1 \ge t_1)$. What is the distribution of T_1?

$$S_1(t_1) = P_r(T_1 \ge t_1) = S(t_1, 0) = \left[e^{\theta(\frac{t_1}{\alpha_1})^{\beta_1}}\right]^{-\frac{1}{\theta}} = e^{-(\frac{t_1}{\alpha_1})^{\beta_1}}$$

$$-\frac{dS_1(t_1)}{dt_1} = f(t_1) = \frac{\beta_1}{\alpha_1}\left(\frac{t_1}{\alpha_1}\right)^{\beta_1 - 1} \cdot e^{-(\frac{t_1}{\alpha_1})^{\beta_1}} \Rightarrow T_1 \sim \cancel{Burr}\ \text{Weib}(\alpha_1, \beta_1)$$

+1 (ii) Calculate the mode-specific hazard function $\lambda_1(t)$ for $C = 1$.

$$\lambda_1(t) = \frac{1}{S(t,t)}\left[-\frac{dS(t_1,t)}{dt_1}\Big|_{t_1=t}\right]$$

$$\frac{dS(t_1,t)}{dt_1}\Big|_{t_1=t} = -\frac{1}{\theta}\left[e^{\theta(\frac{t_1}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]^{-\frac{1}{\theta}-1} \cdot \theta\beta_1\left(\frac{t_1}{\alpha_1}\right)^{\beta_1 - 1} \cdot \frac{1}{\alpha_1} \cdot e^{\theta(\frac{t_1}{\alpha_1})^{\beta_1}}\Big|_{t_1=t}$$

$$= -\left[e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]^{-\frac{1}{\theta}-1} \cdot \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} e^{\theta(\frac{t}{\alpha_1})^{\beta_1}}$$

$$\Rightarrow \lambda_1(t) = \frac{\left[e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]^{-\frac{1}{\theta}-1} \cdot \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} e^{\theta(\frac{t}{\alpha_1})^{\beta_1}}}{\left[e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]^{-\frac{1}{\theta}}} = \frac{\frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} e^{\theta(\frac{t}{\alpha_1})^{\beta_1}}}{e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1}$$

+1 (iii) Calculate the hazard function $h(t)$ for T.

$$h(t) = -\frac{d}{dt}\log S(t)$$

$$S(t) = P(T \ge t) = P(T_1 \ge t, T_2 \ge t) = S(t,t) = \left[e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]^{-\frac{1}{\theta}}$$

$$\Rightarrow \log S(t) = -\frac{1}{\theta}\log\left[e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1\right]$$

$$\Rightarrow h(t) = \frac{1}{\theta} \cdot \frac{\theta \cdot \frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + \theta \cdot \frac{\beta_2}{\alpha_2}\left(\frac{t}{\alpha_2}\right)^{\beta_2 - 1} e^{\theta(\frac{t}{\alpha_2})^{\beta_2}}}{e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1}$$

$$= \frac{\frac{\beta_1}{\alpha_1}\left(\frac{t}{\alpha_1}\right)^{\beta_1 - 1} e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + \frac{\beta_2}{\alpha_2}\left(\frac{t}{\alpha_2}\right)^{\beta_2 - 1} e^{\theta(\frac{t}{\alpha_2})^{\beta_2}}}{e^{\theta(\frac{t}{\alpha_1})^{\beta_1}} + e^{\theta(\frac{t}{\alpha_2})^{\beta_2}} - 1}$$

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2. Lifetimes follow $T_i \sim S(t) = \Pr(T_i \ge t)$, and independent random censoring times follow $C_i \sim G(c) = \Pr(C_i \ge c)$ for $i = 1, \ldots, n$. Let

$$h(t) = -\frac{dS(t)/dt}{S(t)}, \qquad \lambda(c) = -\frac{dG(c)/dc}{G(c)}.$$

We observe censored data $t_i = \min(T_i, C_i)$ and $\delta_i = I(T_i \le C_i)$, $i = 1, \ldots, n$.

$\delta_i = 1 \Rightarrow T_i \le C_i$

+1 (i) Write down the likelihood function using h, S, λ, and G.

$\delta_i = 1$, $\{T_i = t_i, \ C_i \ge t_i\}$

$\Rightarrow L_i = P(T_i = t_i, \ C_i \ge t_i) = f(t_i) \cdot G(t_i) = -\frac{dS(t_i)}{dt_i} \cdot G(t_i) = h(t_i) S(t_i) G(t_i)$

$\delta_i = 0$, $\{T_i > t_i, \ C_i = t_i\}$

$\Rightarrow L_i = P(T_i > t_i, \ C_i = t_i) = S(t_i) \cdot -\frac{dG(t_i)}{dt_i} = S(t_i) \cdot \lambda(t_i) \cdot G(t_i)$

$$\Rightarrow L = \prod_{i=1}^{n} \{h(t_i) S(t_i) G(t_i)\}^{\delta_i} \{S(t_i) \lambda(t_i) G(t_i)\}^{1-\delta_i}$$

$$= \left\{\prod_{i=1}^{n} h(t_i)^{\delta_i} S(t_i)\right\} \times \left\{\prod_{i=1}^{n} G(t_i) \lambda(t_i)^{1-\delta_i}\right\}$$

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+1 (ii) Reduce the likelihood function under the non-informative censoring assumption.

Under the non-informative censoring assumption, we can ignore $\left\{\prod_{i=1}^{n} G(t_i) \lambda(t_i)^{1-\delta_i}\right\}$.

$$\Rightarrow L = \prod_{i=1}^{n} [h(t_i)]^{\delta_i} \cdot S(t_i)$$

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