HW#6 Survival analysis II
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Summary of section2.1 and section2.2 in

Emura T & Chen YH (2014), Gene selection for survival data under dependent censoring, a copula-based approach, *Statistical Methods in Medical Research*, DOI: 10.1177/0962280214533378

Section2.1 is talk about the univariate selection for censored survival data. In univariate selection, a Cox regression based on univariate models

$$h(t \mid x_{ij}) = h_{0j}(t)e^{\beta_j x_{ij}}, \quad j = 1, \cdots, p$$

and there is an assumption

Assumption I: The survival time T and censoring time U are conditionally independent given a gene x_{ij} for all $j = 1, \cdots, p$.

Even when the previous model is incorrect, the univariate estimate $\hat{\beta}_j$ still possesses a valid meaning under Assumption I. This result proved that Assumption I is more important than the previous model in applying univariate selection.

In section2.2, there is a more reasonable assumption is the conditional independent in which T and U are conditionally independent given all components of **x**. And Figure 1 shows an example of how Assumption I fails to hold. Then the result of analysis shows that the conditional independence yields dependency between T and U given only x_j, and thus Assumption I does not hold.

So in general, T and U may be dependent for any given x_j with an unknown dependence structure.

Detail derivation of Eq. (2)

The Laplace transform of a random vector $\mathbf{x}$ is defined as $\varphi(\mathbf{u}) = E\{\exp(-\mathbf{u}'\mathbf{x})\}$.

Hence,

$$\varphi_{\boldsymbol{\beta}_{(-j)},\boldsymbol{\gamma}_{(-j)}}(u,v) = E\{\exp(-ue^{\boldsymbol{\beta}'_{(-j)}\mathbf{x}_{(-j)}})\exp(-ve^{\boldsymbol{\gamma}'_{(-j)}\mathbf{x}_{(-j)}}) \mid x_j\},$$

$$\varphi_{\boldsymbol{\beta}_{(-j)}}(u) = E\{\exp(-ue^{\boldsymbol{\beta}'_{(-j)}\mathbf{x}_{(-j)}}) \mid x_j\} = \varphi_{\boldsymbol{\beta}_{(-j)}\boldsymbol{\gamma}_{(-j)}}(u,0),$$

$$\varphi_{\boldsymbol{\gamma}_{(-j)}}(u) = E\{\exp(-ue^{\boldsymbol{\gamma}'_{(-j)}\mathbf{x}_{(-j)}}) \mid x_j\} = \varphi_{\boldsymbol{\beta}_{(-j)}\boldsymbol{\gamma}_{(-j)}}(0,u)$$

are the Laplace transform for some random vectors.

Then we have

$$\begin{aligned}
& P(T>t, U>u \mid x_j) \\
&= E\{P(T>t, U>u \mid \mathbf{x}) \mid x_j\} \\
&= E\{P(T>t \mid \mathbf{x})P(U>u \mid \mathbf{x}) \mid x_j\} \\
&= E[\exp\{-\Lambda_T(t)e^{\boldsymbol{\beta}'\mathbf{x}}\}\exp\{-\Lambda_U(u)e^{\boldsymbol{\gamma}'\mathbf{x}}\} \mid x_j] \\
&= E[\exp\{-\Lambda_T(t)e^{\beta_j x_j}e^{\boldsymbol{\beta}'_{(-j)}\mathbf{x}_{(-j)}}\}\exp\{-\Lambda_U(u)e^{\gamma_j x_j}e^{\boldsymbol{\gamma}'_{(-j)}\mathbf{x}_{(-j)}}\} \mid x_j] \\
&= \varphi_{\boldsymbol{\beta}_{(-j)},\boldsymbol{\gamma}_{(-j)}}\{\Lambda_T(t)e^{\beta_j x_j}, \Lambda_U(u)e^{\gamma_j x_j}\},
\end{aligned}$$

and then

$$\begin{aligned}
P(T>t \mid x_j) &= E\{P(T>t \mid \mathbf{x}) \mid x_j\} \\
&= E[\exp\{-\Lambda_T(t)e^{\boldsymbol{\beta}'\mathbf{x}}\} \mid x_j] \\
&= E[\exp\{-\Lambda_T(t)e^{\beta_j x_j}e^{\boldsymbol{\beta}'_{(-j)}\mathbf{x}_{(-j)}}\} \mid x_j] \\
&= \varphi_{\boldsymbol{\beta}'_{(-j)}}\{\Lambda_T(t)e^{\beta_j x_j}\}.
\end{aligned}$$

Therefore,

$$\Lambda_T(t)e^{\beta_j x_j} = \varphi^{-1}_{\boldsymbol{\beta}_{(-j)}}\{P(T>t \mid x_j)\}$$

and similarly,

$$\Lambda_U(u)e^{\gamma_j x_j} = \varphi^{-1}_{\boldsymbol{\gamma}_{(-j)}}\{P(U>u \mid x_j)\}.$$

Hence we can obtain

$$P(T>t, U>u \mid x_j) = \varphi_{\boldsymbol{\beta}_{(-j)},\boldsymbol{\gamma}_{(-j)}}[\varphi^{-1}_{\boldsymbol{\beta}_{(-j)}}\{P(T>t \mid x_j)\}, \varphi^{-1}_{\boldsymbol{\gamma}_{(-j)}}\{P(U>u \mid x_j)\}].$$

Detail derivation of Eq. (3)

For the special case of $\boldsymbol{\beta} = \boldsymbol{\gamma}$,

$$
\begin{aligned}
& P(T > t, U > u \mid x_j) \\
&= E\{ P(T > t, U > u \mid \mathbf{x}) \mid x_j \} \\
&= E\{ P(T > t \mid \mathbf{x}) P(U > u \mid \mathbf{x}) \mid x_j \} \\
&= E[\exp\{ -\Lambda_T(t) e^{\boldsymbol{\beta}'\mathbf{x}} \} \exp\{ -\Lambda_U(u) e^{\boldsymbol{\beta}'\mathbf{x}} \} \mid x_j] \\
&= E(\exp[-\{ \Lambda_T(t) + \Lambda_U(u) \} e^{\boldsymbol{\beta}'\mathbf{x}}] \mid x_j) \\
&= E(\exp[-\{ \Lambda_T(t) + \Lambda_U(u) \} e^{\beta_j x_j} e^{\boldsymbol{\beta}'_{(-j)}\mathbf{x}_{(-j)}}] \mid x_j) \\
&= \varphi_{\boldsymbol{\beta}_{(-j)}} [\{ \Lambda_T(t) + \Lambda_U(u) \} e^{\beta_j x_j}].
\end{aligned}
$$

From the previous result we have

$$\Lambda_T(t) e^{\beta_j x_j} = \varphi^{-1}_{\boldsymbol{\beta}_{(-j)}} \{ P(T > t \mid x_j) \}$$

and

$$\Lambda_U(u) e^{\beta_j x_j} = \varphi^{-1}_{\boldsymbol{\beta}_{(-j)}} \{ P(T > t \mid x_j) \}.$$

Hence we can obtain the Archimedean copula representation

$$P(T > t, U > u \mid x_j) = \varphi_{\boldsymbol{\beta}_{(-j)}} [\varphi^{-1}_{\boldsymbol{\beta}_{(-j)}} \{ P(T > t \mid x_j) \} + \varphi^{-1}_{\boldsymbol{\beta}_{(-j)}} \{ P(U > u \mid x_j) \}].$$