HW#4 Survival Analysis II
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1. First we are going to prove the second derivatives of log-partial likelihood function is smaller or equal to 0, for all β. That is:

$$\frac{d^2}{d\beta^2}\log P(\beta) \le 0, \qquad \forall \beta$$

where

$$P(\beta) = \prod_{i=1}^{n}\left\{\frac{\exp(\beta x_i)}{\sum_{\ell \in R(t_i)} \exp(\beta x_\ell)}\right\}^{\delta_i}$$

Since we have $P(\beta)$. We can obtain

$$\log P(\beta) = \sum_{i=1}^{n}\delta_i\left[\beta x_i - \log\left\{\sum_{\ell \in R(t_i)}\exp(\beta x_\ell)\right\}\right]$$

with first and second derivatives

$$\frac{d}{d\beta}\log P(\beta) = \sum_{i=1}^{n}\delta_i\left\{x_i - \frac{\sum_{\ell \in R(t_i)} x_\ell \exp(\beta x_\ell)}{\sum_{\ell \in R(t_i)}\exp(\beta x_\ell)}\right\}$$

$$\frac{d^2}{d\beta^2}\log P(\beta) = \sum_{i=1}^{n}\delta_i\left[-\frac{\sum_{\ell \in R(t_i)} x_\ell^{\,2}\exp(\beta x_\ell)\sum_{\ell \in R(t_i)}\exp(\beta x_\ell) - \sum_{\ell \in R(t_i)} x_\ell \exp(\beta x_\ell)\sum_{\ell \in R(t_i)} x_\ell \exp(\beta x_\ell)}{\left\{\sum_{\ell \in R(t_i)}\exp(\beta x_\ell)\right\}^2}\right]$$

Now, we consider

$$\sum_{\ell \in R(t_i)} x_\ell^{\,2}\exp(\beta x_\ell)\sum_{\ell \in R(t_i)}\exp(\beta x_\ell) - \sum_{\ell \in R(t_i)} x_\ell \exp(\beta x_\ell)\sum_{\ell \in R(t_i)} x_\ell \exp(\beta x_\ell)$$

this can be expand as follow

$$\begin{aligned}&\{x_1^2\exp(\beta x_1)+\cdots+x_k^2\exp(\beta x_k)\}\cdot\{\exp(\beta x_1)+\cdots+\exp(\beta x_k)\}\\ -&\{x_1\exp(\beta x_1)+\cdots+x_k\exp(\beta x_k)\}\cdot\{x_1\exp(\beta x_1)+\cdots+x_k\exp(\beta x_k)\}\end{aligned}$$

here we can assume $\{1,\cdots,k\} = \ell \in R(t_i)$.

therefore,

$$\begin{aligned}
& x_1^2 \exp(2\beta x_1) + x_1^2 \exp\{\beta(x_1+x_2)\} + \cdots + x_1^2 \exp\{\beta(x_1+x_k)\} \\
& + x_2^2 \exp\{\beta(x_1+x_2)\} + x_2^2 \exp(2\beta x_2) + \cdots + x_2^2 \exp\{\beta(x_2+x_k)\} \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& + x_k^2 \exp\{\beta(x_1+x_k)\} + x_k^2 \exp\{\beta(x_2+x_k)\} + \cdots + x_k^2 \exp(2\beta x_k) \\
& - x_1^2 \exp(2\beta x_1) - x_1 x_2 \exp\{\beta(x_1+x_2)\} - \cdots - x_1 x_k \exp\{\beta(x_1+x_k)\} \\
& - x_1 x_2 \exp\{\beta(x_1+x_2)\} - x_2^2 \exp(2\beta x_2) - \cdots - x_2 x_k \exp\{\beta(x_2+x_k)\} \\
& \qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad \vdots \\
& - x_1 x_k \exp\{\beta(x_1+x_k)\} - x_2 x_k \exp\{\beta(x_2+x_k)\} - \cdots - x_k^2 \exp(2\beta x_k)
\end{aligned}$$

some terms can be canceled and the remaining terms can be write as

$$\sum_{a<b} (x_a - x_b)^2 \exp\{\beta(x_a+x_b)\}, \qquad \forall a,b \in \ell$$

Then we have

$$\frac{d^2}{d\beta^2}\log P(\beta) = \sum_{i=1}^{n} \delta_i \left[-\frac{\sum_{a<b} (x_a-x_b)^2 \exp\{\beta(x_a+x_b)\}}{\left\{ \sum_{\ell \in R(t_i)} \exp(\beta x_\ell) \right\}^2} \right], \qquad \forall a,b \in \ell$$

since both

$$\sum_{a<b} (x_a - x_b)^2 \exp\{\beta(x_a+x_b)\} \geq 0, \qquad \forall a,b \in \ell$$

and

$$\left\{ \sum_{\ell \in R(t_i)} \exp(\beta x_\ell) \right\}^2 \geq 0$$

are true for all β. And δ_i is the censoring indicator.

Hence we have proved that

$$\frac{d^2}{d\beta^2}\log P(\beta) \leq 0, \qquad \forall \beta$$

2. During class, we have proved that

$$\hat{\beta} = \log \frac{\sum_{x_i=1} W_{\hat{\beta}}(t_i)\delta_i / \overline{Y_1}(t_i)}{\sum_{x_i=0} W_{\hat{\beta}}(t_i)\delta_i / \overline{Y_0}(t_i)},$$

where

$$\overline{Y_0}(t_i) = \sum_{\substack{\ell \in R(t_i) \\ x_i=0}} 1, \ \ \overline{Y_1}(t_i) = \sum_{\substack{\ell \in R(t_i) \\ x_i=1}} 1 \ \text{ and } \ W_{\hat{\beta}}(t_i) = \frac{\overline{Y_0}(t_i)\overline{Y_1}(t_i)}{\overline{Y_0}(t_i) + e^{\beta}\overline{Y_1}(t_i)}.$$

Then we use the data in example 1. We can obtain

$$\overline{Y_0}(t_1) = \sum_{\substack{\ell \in R(t_1) \\ x_i=0}} 1 = 2, \ \ \overline{Y_1}(t_1) = \sum_{\substack{\ell \in R(t_1) \\ x_i=1}} 1 = 2,$$

$$\overline{Y_0}(t_3) = \sum_{\substack{\ell \in R(t_3) \\ x_i=0}} 1 = 2, \ \ \overline{Y_1}(t_3) = \sum_{\substack{\ell \in R(t_3) \\ x_i=1}} 1 = 3,$$

$$\overline{Y_0}(t_5) = \sum_{\substack{\ell \in R(t_5) \\ x_i=0}} 1 = 1, \ \ \overline{Y_1}(t_5) = \sum_{\substack{\ell \in R(t_5) \\ x_i=1}} 1 = 1,$$

$$W_{\hat{\beta}}(t_1) = \frac{4}{2+2e^{\beta}}, \ \ W_{\hat{\beta}}(t_3) = \frac{6}{2+3e^{\beta}}, \ \ W_{\hat{\beta}}(t_5) = \frac{1}{1+e^{\beta}}.$$

Since $i = 2,4$ have been censored, so we don't need to compute.

Therefore, by the formula we have

$$\hat{\beta} = \log \frac{\sum_{x_i=1} W_{\hat{\beta}}(t_i)\delta_i / \overline{Y_1}(t_i)}{\sum_{x_i=0} W_{\hat{\beta}}(t_i)\delta_i / \overline{Y_0}(t_i)}$$

$$= \log \left(\frac{\frac{2}{2+e^{\hat{\beta}}} + \frac{2}{2+3e^{\hat{\beta}}}}{\frac{1}{1+e^{\hat{\beta}}}} \right)$$

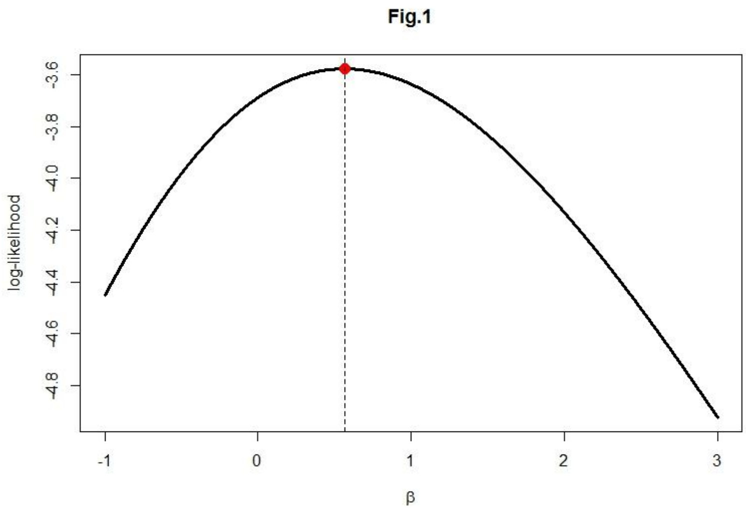
3. By the previous result, we have

$$\hat{\beta} = \log\left(\frac{\frac{2}{2+e^{\hat{\beta}}} + \frac{2}{2+3e^{\hat{\beta}}}}{\frac{1}{1+e^{\hat{\beta}}}} \right)$$

Therefore, we can perform recursive algorithm by R to approach $\hat{\beta}$.

The result is $\hat{\beta} = 0.5643507$.

Now, we can plot log-partial likelihood function to check whether it is PMLE or not.



Hence the point in Fig.1 is the PMLE. The result from recursive algorithm is correct.

R code

```
########## beta iterative function ##########


it_func=function (beta) {


    b1=2/(2+2*exp(beta))

    b2=2/(2+3*exp(beta))

    b3=1/(1+exp(beta))

    log((b1+b2)/b3)


}


########## iteration ##########

beta=1

repeat{


    beta_hat=it_func(beta)

    if (abs(beta-beta_hat)<10^-6)

        break

    else

        beta=beta_hat


}

beta_hat
```

```
########## Log-partial likelihood function ##########
ll_func=function (beta) {

   b1=2+3*exp(beta)
   b2=2+2*exp(beta)
   b3=1+exp(beta)
   2*beta-log(b1)-log(b2)-log(b3)

}

############### Plot log-partial likelihood function ###############
q=seq(-1,3,by=0.0001)
ll=c()

for(i in 1:length(q)){

   beta=q[i]
   ll[i]=ll_func(beta)

}

plot(q,ll,type="l",xlab = expression(beta),ylab = "log-likelihood",main="Fig.1",lwd =
3)
points(beta_hat,max(ll),cex=1.5,col=2,pch=16)
abline(v=beta_hat,lty=2.5,lwd=1.5)
```