Statistical Inference II

Mid-term exam: 2013/4/16 (Tue)

Q1
Q2
Q3

YOUR NAME_______________________

NOTE1: Please write down the derivation of your answer very clearly for all questions. The score will be reduced when you only write answer. Also, the score will be reduced if the derivation is not clear. The score will be added even when your answer is incorrect but the derivation is correct.

1. Let $X_1,\dots,X_n \overset{iid}{\sim} f_{\mu,\sigma^2}(x)$, where

$$f_{\mu,\sigma^2}(x) = \sqrt{\frac{2}{\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} I(x \ge \mu)$$

is a left-truncated normal distribution, truncated at unknown value $\mu \in R$.

1) Find the MLE under $H_0 : \mu = 0$.

2) Find the MLE $\hat{\theta} = (\hat{\mu}, \hat{\sigma}^2)$.

3) Find the likelihood ratio $\lambda(X)$ for $H_0 : \mu = 0$ vs. $H_0 : \mu \neq 0$.

4) Find the form of the likelihood ratio test using some cut off value *c*.

2. Let $(X_1,\ldots,X_5)=(5,6,7,2,4)$ be a realization from iid discrete uniform distribution on $\{1,2,\ldots,\theta\}$ where $\theta \geq 2$ is an integer.

(a) Write down the likelihood function and then find the MLE $\hat{\theta}$.

(b) Find the likelihood ratio (LR) test for $H_0:\theta \leq \theta_0$ vs. $H_1:\theta > \theta_0$ and show that the size is zero (i.e., $\sup_{\theta\leq\theta_0} P_\theta$(Reject H_0) =0).

(c) Find the LR test for $H_0:\theta=\theta_0$ vs. $H_1:\theta \neq \theta_0$ with Type I error α.

(d) Test $H_0:\theta=8$ vs. $H_1:\theta \neq 8$ with α =1/32.

(e) In this example, neither the Wald test nor Score test for $H_0:\theta=\theta_0$ is applicable. Give a reason.

3. Let $(X_{j1}, X_{j2}),\ j \in \{1, \ldots, n\}$ be a realization from iid bivariate normal distribution

$$f_{\boldsymbol{\mu}, \boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{2\pi |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{ -\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})' \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right\},$$

where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and the parameters $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_{22}^2 \end{bmatrix}$ are unknown.

Also, let

$$W = \frac{\sum_j (X_{j1} - \bar{X}_1)(X_{j2} - \bar{X}_2)}{\sqrt{\sum_j (X_{j1} - \bar{X}_1)^2 \sum_j (X_{j2} - \bar{X}_2)^2}}$$

be the estimator of the correlation coefficient. Consider testing

$$H_0 : \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}} = 0 \quad \text{vs.} \quad H_1 : \frac{\sigma_{12}}{\sigma_{11}\sigma_{22}} \neq 0.$$

1) Find the MLEs under $H_0 \cup H_1$ and under H_0, respectively (only answer).
2) Find the maximized likelihoods under $H_0 \cup H_1$ and H_0, respectively.
3) Write the likelihood ratio statistic in terms of W.
4) According to the LR test, we reject H_0 when $|W| > c$. Find the critical value based on the asymptotic approximation.

Answers: This is a simplified answer. In the exam, you need to write more detailed calculations.

<u>Answer 1</u>

1) $\hat{\mu}_0 = 0$, $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^{\ 2}$.

$\because$) Under $H_0 : \mu = 0$, $\ell(0,\sigma^2) = -\frac{1}{2}\log\left(\frac{2}{\pi}\right) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n} X_i^{\ 2}$.

Solving $\frac{\partial \ell(0,\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n} X_i^{\ 2} = 0$,

we have $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^{\ 2}$.

2) $\hat{\mu} = X_{(1)}$, $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})^2$.

$\because$) Let $\ell(\mu,\sigma^2) = -\frac{1}{2}\log\left(\frac{2}{\pi}\right) - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i - \mu)^2$. Then,

$\ell(\mu,\sigma^2) \le \ell(X_{(1)},\sigma^2)$ for any σ^2.

Solving $\frac{\partial \ell(X_{(1)},\sigma^2)}{\partial \sigma^2} = -\frac{n}{2\sigma^2} - \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n}(X_i - X_{(1)})^2 = 0$,

$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - X_{(1)})^2$.

3) $\lambda(X) = \left(\frac{\hat{\sigma}^2}{\hat{\sigma}_0^2}\right)^{n/2} I(X_{(1)} \ge 0)$

4) $\lambda(X) < c \Leftrightarrow \frac{\hat{\sigma}^2}{\hat{\sigma}_0^2} < c_1$ or $X_{(1)} < 0$ for some c_1.

<u>Answer 2</u>

a) $\ell(\theta) = \theta^{-5} I(X_{(5)} \le \theta)$, hence $\hat{\theta} = X_{(5)} = 7$.

b) Since $\max_{\theta=2,3,\ldots} \ell(\theta) = X_{(5)}^{\ -5}$ and $\max_{\theta=2,3,\ldots\theta_0} \ell(\theta) = X_{(5)}^{\ -5} I(X_{(5)} \le \theta_0)$,

$LR = \frac{\max_{\theta=2,3,\ldots\theta_0} \ell(\theta)}{\max_{\theta=2,3,\ldots} \ell(\theta)} = I(X_{(5)} \le \theta_0)$. We reject $H_0 : \theta \le \theta_0$ when $X_{(5)} > \theta_0$,

which has the size $P_\theta(X_{(5)} > \theta_0) = 0 \quad \forall \theta \le \theta_0$.

c) $LR=\frac{\ell(\theta_0)}{\max_{\theta=2,3,...}\ell(\theta)}=(X_{(5)}/\theta_0)^5 I(X_{(5)}\le\theta_0)<c$ iff $X_{(5)}<\tilde{c}$ or $X_{(5)}>\theta_0$.

Solving $\alpha=P_{\theta_0}(X_{(5)}<\tilde{c}$ or $X_{(5)}>\theta_0)=P_{\theta_0}(X_{(5)}<\tilde{c})=(\tilde{c}/\theta_0)^5$, $\tilde{c}=\theta_0\alpha^{1/5}$.

We reject $H_0:\theta=\theta_0$ when $X_{(5)}<\theta_0\alpha^{1/5}$ or $X_{(5)}>\theta_0$.

d) $X_{(5)}=7>$ 8*(1/32)^(1/5)=4. Accept the null hypothesis.

e) The likelihood function is not differentiable. Hence, the score function and Fisher information matrix are not obtained.

Answer 3

1) $\hat{\boldsymbol{\mu}}=\hat{\boldsymbol{\mu}}_0=\frac{1}{n}\sum_{j=1}^{n}\mathbf{x}_j$,

$$\hat{\boldsymbol{\Sigma}}=\frac{1}{n}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'=\begin{bmatrix}S_1^2 & S_{12}\\ S_{12} & S_2^2\end{bmatrix},\quad \hat{\boldsymbol{\Sigma}}_0=\frac{1}{n}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'=\begin{bmatrix}S_1^2 & 0\\ 0 & S_2^2\end{bmatrix}.$$

2) Since

$$\ell(\boldsymbol{\mu},\boldsymbol{\Sigma})=\prod_{j=1}^{n}f_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}_j)$$

$$=\frac{1}{(2\pi)^n|\boldsymbol{\Sigma}|^{n/2}}\exp\left\{-\frac{1}{2}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})'\boldsymbol{\Sigma}^{-1}(\mathbf{x}_j-\boldsymbol{\mu})\right\},$$

$$=\frac{1}{(2\pi)^n|\boldsymbol{\Sigma}|^{n/2}}\exp\left\{-\frac{1}{2}\mathrm{tr}\left[\boldsymbol{\Sigma}^{-1}\sum_{j=1}^{n}(\mathbf{x}_j-\boldsymbol{\mu})(\mathbf{x}_j-\boldsymbol{\mu})'\right]\right\}$$

we have $\ell(\hat{\boldsymbol{\mu}},\hat{\boldsymbol{\Sigma}})=\frac{\exp(-n)}{(2\pi)^n|\hat{\boldsymbol{\Sigma}}|^{n/2}}$ and $\ell(\hat{\boldsymbol{\mu}}_0,\hat{\boldsymbol{\Sigma}}_0)=\frac{\exp(-n)}{(2\pi)^n|\hat{\boldsymbol{\Sigma}}_0|^{n/2}}$.

3) $\ell(\hat{\boldsymbol{\mu}}_0,\hat{\boldsymbol{\Sigma}}_0)=\left(\frac{|\hat{\boldsymbol{\Sigma}}_0|}{|\hat{\boldsymbol{\Sigma}}|}\right)^{n/2}=(1-W^2)^{n/2}$.

4) $c=\sqrt{1-\exp\{-\chi^2_{df=1}(1-\alpha)/n\}}$.