Statistical Inference III

Midterm exam I: **[+32 points]**

2016/11/7 (Mon) 14:00-16:50

Q1
Q2
Q3
Q4

YOUR NAME______Jia-Han Shih______

NOTE1: Please write down the derivation of your answer very clearly for all questions. The score will be reduced if you only write the answer or if the derivation is not clear. The score will be given even when your answer has a minor mistake but the derivations are clearly stated.

1. [+10] Exponential family

Let $p_\theta(x) = C(\theta)\exp\left[\sum_{j=1}^{k}\theta_j T_j(x)\right]h(x)$, $x \in \aleph$, be a density w.r.t. a σ-finite measure μ.

(1)[+1] Define the *natural parameter space*.

<u>Answer:</u>

The natural parameter space is

$$\Omega = \left\{\theta = (\theta_1, \theta_2, \cdots, \theta_k) : \int_\chi \exp\left[\sum_{j=1}^{k}\theta_j T_j(x)\right]h(x)d\mu(x) < \infty\right\}.$$

(2) [+1] State the definition that a set Ω is *convex*.

<u>Answer:</u>

Ω is convex if and only if for all $\theta, \theta' \in \Omega$ and $0 < \alpha < 1$ imply $\alpha\theta + (1-\alpha)\theta' \in \Omega$.

(3) [+3] Show that the natural parameter space is convex.

<u>Answer:</u>

For all $\theta, \theta' \in \Omega$ and let $h(x)d\mu(x) = d\mu_h(x)$, we have

$$\begin{aligned}
&\int_\chi \exp\left[\sum_{j=1}^{k}\{\alpha\theta_j + (1-\alpha)\theta'_j\}T_j(x)\right]d\mu_h(x) \\
&= \int_\chi \exp\left[\alpha\sum_{j=1}^{k}\theta_j T_j(x)\right]\exp\left[(1-\alpha)\sum_{j=1}^{k}\theta'_j T_j(x)\right]d\mu_h(x) \\
&\le \left\{\int_\chi \exp\left[\alpha\sum_{j=1}^{k}\theta_j T_j(x)\right]^{\frac{1}{\alpha}} d\mu_h(x)\right\}^{\alpha} \\
&\qquad \times \left\{\int_\chi \exp\left[(1-\alpha)\sum_{j=1}^{k}\theta'_j T_j(x)\right]^{\frac{1}{1-\alpha}} d\mu_h(x)\right\}^{1-\alpha} < \infty.
\end{aligned}$$

The inequality follows from the Hölder's inequality since

$$\alpha + (1-\alpha) = 1 \quad \text{and} \quad \frac{1}{\alpha}, \frac{1}{1-\alpha} > 1.$$

Hence we have shown that the natural parameter space is convex. □

(4) [+5] For any function ϕ, prove

$$\frac{\partial}{\partial\theta_1}\int_\chi \phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x) = \int_\chi \frac{\partial}{\partial\theta_1}\phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x).$$

Answer:

We first rewrite the integral as follows

$$\int_\chi \phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x)$$
$$= \int_\chi e^{\theta_1 T_1(x)}\phi(x)\exp\left[\sum_{j=2}^k \theta_j T_j(x)\right]h(x)d\mu(x) = \int_\chi e^{\theta_1 T_1(x)}d\mu_1(x),$$

where

$$\phi(x)\exp\left[\sum_{j=2}^k \theta_j T_j(x)\right]h(x)d\mu(x) = d\mu_1(x).$$

Now, let

$$\psi(\theta_1) = \int_\chi e^{\theta_1 T_1(x)}d\mu_1(x).$$

For all fixed θ_1^0, we have

$$\frac{\partial}{\partial\theta_1}\int_\chi \phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x)\bigg|_{\theta_1=\theta_1^0} = \frac{\partial}{\partial\theta_1}\psi(\theta_1)\bigg|_{\theta_1=\theta_1^0}$$
$$= \lim_{\theta_1\to\theta_1^0}\frac{\psi(\theta_1)-\psi(\theta_1^0)}{\theta_1-\theta_1^0} = \lim_{n\to\infty}\frac{\psi(\theta_1^{(n)})-\psi(\theta_1^0)}{\theta_1^{(n)}-\theta_1^0},$$

where $\theta_1^{(n)}\to\theta_1^0$ as $n\to\infty$, and

$$\int_\chi \frac{\partial}{\partial\theta_1}\phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)\bigg|_{\theta_1=\theta_1^0} d\mu(x) = \int_\chi \frac{\partial}{\partial\theta_1}e^{\theta_1 T_1(x)}\bigg|_{\theta_1=\theta_1^0} d\mu_1(x)$$
$$= \int_\chi T_1(x)e^{\theta_1^0 T_1(x)}d\mu_1(x).$$

Therefore, prove the equation

$$\frac{\partial}{\partial\theta_1}\int_\chi \phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x) = \int_\chi \frac{\partial}{\partial\theta_1}\phi(x)\exp\left[\sum_{j=1}^k \theta_j T_j(x)\right]h(x)d\mu(x)$$

is equivalent to prove

$$\lim_{n\to\infty}\frac{\psi(\theta_1^{(n)})-\psi(\theta_1^0)}{\theta_1^{(n)}-\theta_1^0} = \int_\chi T_1(x)e^{\theta_1^0 T_1(x)}d\mu_1(x),$$

where $\theta_1^{(n)}\to\theta_1^0$ as $n\to\infty$.

Since

$$\frac{\psi(\theta_1)-\psi(\theta_1^0)}{\theta_1-\theta_1^0}=\frac{\int_\chi e^{\theta_1 T_1(x)}d\mu_1(x)-\int_\chi e^{\theta_1^0 T_1(x)}d\mu_1(x)}{\theta_1-\theta_1^0}=\int_\chi \frac{e^{\theta_1 T_1(x)}-e^{\theta_1^0 T_1(x)}}{\theta_1-\theta_1^0}d\mu_1(x)\,.$$

Consider the inequality

$$\left|\frac{e^{az}-1}{z}\right|\le\frac{e^{\delta|a|}}{\delta},\qquad |z|\le\delta\,.$$

The proof of the inequality is given in the end of this question. Therefore, for all $|\theta_1-\theta_1^0|\le\delta$, we obtain

$$\left|\frac{e^{\theta_1 T_1(x)}-e^{\theta_1^0 T_1(x)}}{\theta_1-\theta_1^0}\right|=e^{\theta_1^0 T_1(x)}\left|\frac{e^{(\theta_1-\theta_1^0)T_1(x)}-1}{\theta_1-\theta_1^0}\right|\le e^{\theta_1^0 T_1(x)}\frac{e^{\delta|T_1(x)|}}{\delta}$$
$$\le\frac{1}{\delta}\{e^{(\theta_1^0+\delta)T_1(x)}+e^{(\theta_1^0-\delta)T_1(x)}\}.$$

Let $\theta_1^{(n)}$ be a sequence such that $\theta_1^{(n)}\to\theta_1^0$ as $n\to\infty$, that is, there exists $\delta>0$, $n_0\in\mathrm{N}$ such that $|\theta_1-\theta_1^0|\le\delta$ for all $n\ge n_0$. Thus,

$$|f_n(x)|\equiv\left|\frac{e^{\theta_1^{(n)}T_1(x)}-e^{\theta_1^0 T_1(x)}}{\theta_1^{(n)}-\theta_1^0}\right|\le\frac{1}{\delta}\{e^{(\theta_1^0+\delta)T_1(x)}+e^{(\theta_1^0-\delta)T_1(x)}\}\equiv g(x)$$

and

$$\int_\chi g(x)d\mu_1(x)=\frac{1}{\delta}\left\{\int_\chi\phi(x)\exp\left[(\theta_1+\delta)T_1(x)+\sum_{j=2}^k\theta_jT_j(x)\right]h(x)d\mu(x)\right.$$
$$\left.+\int_\chi\phi(x)\exp\left[(\theta_1-\delta)T_1(x)+\sum_{j=2}^k\theta_jT_j(x)\right]h(x)d\mu(x)\right\}<\infty.$$

By the dominated convergence theorem

$$\lim_{n\to\infty}\int_\chi f_n(x)d\mu_1(x)=\int_\chi\lim_{n\to\infty}f_n(x)d\mu_1(x)$$
$$\Leftrightarrow\lim_{n\to\infty}\int_\chi\frac{e^{\theta_1^{(n)}T_1(x)}-e^{\theta_1^0T_1(x)}}{\theta_1^{(n)}-\theta_1^0}d\mu_1(x)=\int_\chi\lim_{n\to\infty}\frac{e^{\theta_1^{(n)}T_1(x)}-e^{\theta_1^0T_1(x)}}{\theta_1^{(n)}-\theta_1^0}d\mu_1(x)$$
$$\Leftrightarrow\lim_{n\to\infty}\frac{\psi(\theta_1^{(n)})-\psi(\theta_1^0)}{\theta_1^{(n)}-\theta_1^0}=\int_\chi T_1(x)e^{\theta_1^0T_1(x)}d\mu_1(x).$$

Hence we have proven the desired result. □

Proof of the inequality

$$\left|\frac{e^{az}-1}{z}\right| \le \frac{e^{\delta|a|}}{\delta}, \qquad \text{for all } |z| \le \delta .$$

Proof.

For all $|z| \le \delta$, and since exponential function can be defined by power series, we have

$$\begin{aligned}
\left|\frac{e^{az}-1}{z}\right| &= \left|\frac{1}{z}\left\{\sum_{i=0}^{\infty}\frac{(az)^i}{i!}-1\right\}\right| = \left|\frac{1}{z}\left\{1+\sum_{i=1}^{\infty}\frac{(az)^i}{i!}-1\right\}\right| = \left|\frac{1}{z}\sum_{i=1}^{\infty}\frac{a^i z^i}{i!}\right| = \left|\sum_{i=1}^{\infty}\frac{a^i z^{i-1}}{i!}\right| \\
&\le \sum_{i=1}^{\infty}\frac{|a|^i |z|^{i-1}}{i!} \le \sum_{i=1}^{\infty}\frac{|a|^i \delta^{i-1}}{i!} = \frac{1}{\delta}\sum_{i=1}^{\infty}\frac{|a|^i \delta^i}{i!} = \frac{e^{\delta|a|}-1}{\delta} \\
&\le \frac{e^{\delta|a|}}{\delta}.
\end{aligned}$$

Then we finished the proof. □

2. [+10] MP test

Let X be a random variable with $p(x) = P(X = x)$, $x \in \aleph = (1, 2, 3, 4, 5, 6)$.

Consider a test for $H_0 : p(x) = \frac{1}{6}$ vs. $H_1 : p(x) = \frac{x}{21}$.

(1) [+2] Derive a MP test with level $\alpha = 2/5$.

Answer:

Consider the critical function

$$\phi(x) = \begin{cases} 1 & \text{if } x = 5, 6, \\ \frac{2}{5} & \text{if } x = 4, \\ 0 & \text{if } x = 1, 2, 3. \end{cases}$$

The expectation under the null hypothesis is

$$\begin{aligned} E_0\{\phi(X)\} &= 1 \times \Pr_0(X = 5 \text{ or } 6) + \frac{2}{5} \times \Pr_0(X = 4) \\ &= 1 \times \left(\frac{1}{6} + \frac{1}{6}\right) + \frac{2}{5} \times \frac{1}{6} = \frac{1}{3} + \frac{1}{15} \\ &= \frac{2}{5}. \end{aligned}$$

Hence $\phi(x)$ is a MP test of level $\alpha = 2/5$.

(2) [+2] Calculate the power of the above test.

Answer:

The power is the expectation of the critical function under the alternative hypothesis

$$\begin{aligned} E_1\{\phi(X)\} &= 1 \times \Pr_1(X = 5 \text{ or } 6) + \frac{2}{5} \times \Pr_1(X = 4) \\ &= 1 \times \left(\frac{5}{21} + \frac{6}{21}\right) + \frac{2}{5} \times \frac{4}{21} = \frac{55}{105} + \frac{8}{105} \\ &= \frac{3}{5}. \end{aligned}$$

(3) [+2] Derive a MP test with level $\alpha = i/6$, $i \in (1, 2, 3, 4, 5, 6)$.

Answer:

Consider the critical function

$$\phi(x) = \begin{cases} 1 & \text{if } x > 6 - i, \\ 0 & \text{if } x \le 6 - i. \end{cases}$$

The expectation under the null hypothesis is

$$E_0\{\phi(X)\} = \Pr_0(X > 6 - i) = i/6, \qquad i = 1, 2, \cdots, 6.$$

This can be easily seen as follows:

$$\text{for } i = 1, \Pr_0(X > 5) = \Pr_0(X = 6) = 1/6,$$

$$\text{for } i = 2, \Pr_0(X > 4) = \Pr_0(X = 5 \text{ or } 6) = 2/6,$$

$$\vdots$$

(4)[+4] Derive a power function of the above test (as a function of *i*)

Answer:

The power function of the test above is

$$E_1\{\phi(X)\} = \Pr_1(X > 6 - i).$$

By observing the following patterns

$$\text{for } i = 1, \Pr_1(X > 5) = \Pr_1(X = 6) = \frac{6}{21} = \frac{6-0}{21},$$

$$\text{for } i = 2, \Pr_1(X > 4) = \Pr_1(X = 5 \text{ or } 6) = \frac{5}{21} + \frac{6}{21} = \frac{6-1}{21} + \frac{6-0}{21} = \frac{6 \times 2 - (1+0)}{21},$$

$$\text{for } i = 3, \Pr_1(X > 3) = \Pr_1(X = 4 \text{ or } 5 \text{ or } 6) = \frac{4}{21} + \frac{5}{21} + \frac{6}{21} = \frac{6-2}{21} + \frac{6-1}{21} + \frac{6-0}{21} = \frac{6 \times 3 - (2+1+0)}{21},$$

$$\vdots$$

Therefore, the formula for general i is

$$\Pr_1(X > 6 - i) = \frac{6 \times i - \frac{(i-1+0) \times i}{2}}{21} = \frac{13i - i^2}{42} = \frac{i(13 - i\}}{42}.$$

Hence we have derived the formula of the power function.

3. [+4] The Neyman-Peason Lemma

Let $X \sim \mathrm{P} = \{P_0, P_1\}$ and p_i be the p.d.f. of P_i, $i = 1,2$ w.r.t. a common σ-finite measure. Consider a problem of testing $H_0 : P_0$ vs. $H_1 : P_1$ with level α. Construct a test ϕ that satisfies

(i) $0 \le \phi(x) \le 1$, (ii) $E_0[\phi(X)] = \alpha$, (iii) ϕ is most powerful among all level α tests.

[with proofs of (i)-(iii)]

Answer:

Let $\alpha(c) = \Pr_0\{ p_1(X)/p_0(X) > c \}$ hence $\alpha(c)$ is a survival function of random variable $p_1(X)/p_0(X)$ and it is non-increasing and right continuous. Therefore, for all $0 < \alpha < 1$, there always exists $c_0 > c_0^-$ satisfying the inequality $\alpha(c_0) \le \alpha \le \alpha(c_0^-)$. Thus, consider the test

$$\phi(x) = \begin{cases} 1 & \text{if } \dfrac{p_1(x)}{p_0(x)} > c, \\ \dfrac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} & \text{if } \dfrac{p_1(x)}{p_0(x)} = c, \\ 0 & \text{if } \dfrac{p_1(x)}{p_0(x)} < c, \end{cases}$$

where $\alpha(c_0) \le \alpha \le \alpha(c_0^-)$. Then the test ϕ satisfies (i)-(iii).

Proof of (i).

$$\alpha(c_0) \le \alpha \le \alpha(c_0^-) \Rightarrow 0 \le \alpha - \alpha(c_0) \le \alpha(c_0^-) - \alpha(c_0) \Rightarrow 0 \le \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} \le 1.$$

Therefore, $0 \le \phi(x) \le 1$ is satisfied.

Proof of (ii).

$$\begin{aligned} E_0\{\phi(X)\} &= 1 \times \Pr_0\left\{ \frac{p_1(x)}{p_0(x)} > c \right\} + \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} \times \Pr_0\left\{ \frac{p_1(x)}{p_0(x)} = c \right\} \\ &= \alpha(c_0) + \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} \times \left[\Pr_0\left\{ \frac{p_1(x)}{p_0(x)} \le c \right\} - \Pr_0\left\{ \frac{p_1(x)}{p_0(x)} < c \right\} \right] \\ &= \alpha(c_0) + \frac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} \times \{ \alpha(c_0^-) - \alpha(c_0) \} = \alpha(c_0) + \alpha - \alpha(c_0) \\ &= \alpha. \end{aligned}$$

Therefore, $E_0\{\phi(X)\} = \alpha$ is satisfied.

Proof of (iii).

Suppose ϕ^* is a test satisfying $E_0\{\phi(X)\}\le\alpha$. Now, consider two sets $S^+=\{x:\phi(x)-\phi^*(x)>0\}$ and $S^-=\{x:\phi(x)-\phi^*(x)<0\}$. Then we have the following two conclusions:

If $x\in S^+\Rightarrow\phi(x)-\phi^*(x)>0\Rightarrow\phi(x)>0\Rightarrow p_1(x)\ge c_0p_0(x)$.

If $x\in S^-\Rightarrow\phi(x)-\phi^*(x)<0\Rightarrow\phi(x)<0\Rightarrow p_1(x)\le c_0p_0(x)$.

Consider the integral

$$\begin{aligned}&\int_\chi\{\phi(x)-\phi^*(x)\}\{p_1(x)-c_0p_0(x)\}d\mu(x)\\&=\int_{S^+\cup S^-}\{\phi(x)-\phi^*(x)\}\{p_1(x)-c_0p_0(x)\}d\mu(x)\\&=\int_{S^+}\underbrace{\{\phi(x)-\phi^*(x)\}}_{\ge0}\underbrace{\{p_1(x)-c_0p_0(x)\}}_{>0}d\mu(x)\\&\quad+\int_{S^-}\underbrace{\{\phi(x)-\phi^*(x)\}}_{\le0}\underbrace{\{p_1(x)-c_0p_0(x)\}}_{<0}d\mu(x)\ge0.\end{aligned}$$

Then we have

$$\begin{aligned}&\int_\chi\{\phi(x)-\phi^*(x)\}\{p_1(x)-c_0p_0(x)\}d\mu(x)\ge0\\&\Rightarrow\int_\chi\{\phi(x)-\phi^*(x)\}p_1(x)d\mu(x)\ge c_0\int_\chi\{\phi(x)-\phi^*(x)\}p_0(x)d\mu(x)\\&\qquad=c_0[E_0\{\phi(X)\}-E_0\{\phi^*(X)\}]\\&\qquad=c_0[\alpha-E_0\{\phi^*(X)\}]\ge0.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\int_\chi\{\phi(x)-\phi^*(x)\}p_1(x)d\mu(x)\ge0\\&\Rightarrow E_1\{\phi(X)\}\ge E_1\{\phi^*(X)\}.\end{aligned}$$

Therefore, we have shown that ϕ is most powerful among level α test. Hence we have construct a test ϕ satisfying (i)-(iii) with proofs. □

[+8] Monotone likelihood ratio (MLR)

(i) [+2] Define the hyper-geometric (HG) distribution.

Answer:

Let X follows the HG distribution with probability mass function

$$p_D(x) = \Pr_D(X = x) = \frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}},$$

where $\max(0, n-N+D) \le x \le \min(n, D)$.

(ii) [+2] Show that the HG distribution gives MLR.

Answer:

Consider the hypothesis testing

$$H_0 : D \le D_0 \quad \text{versus} \quad H_1 : D > D_0.$$

For $K \in \mathrm{N}$, the likelihood ratio is

$$\frac{p_{D+K}(x)}{p_D(x)} = \frac{p_{D+K}(x)}{p_{D+K-1}(x)} \times \frac{p_{D+K-1}(x)}{p_{D+K-2}(x)} \times \cdots \times \frac{p_{D+1}(x)}{p_D(x)}.$$

Therefore, it is sufficient to verify

$$\frac{p_{D+1}(x)}{p_D(x)} = \frac{\binom{D+1}{x}\binom{N-D-1}{n-x}}{\binom{D}{x}\binom{N-D}{n-x}} = \frac{\frac{(D+1)!}{x!(D+1-x)!} \cdot \frac{(N-D-1)!}{(n-x)!(N-D-1-n+x)!}}{\frac{D!}{x!(D-x)!} \cdot \frac{(N-D)!}{(n-x)!(N-D-n+x)!}}$$

$$= \frac{D+1}{N-D} \cdot \frac{N-D-n+x}{D+1-x}.$$

Then we obtain

$$\frac{p_{D+1}(x)}{p_D(x)} = \begin{cases} 0 & \text{if } x = n-N+D, \\ \frac{D+1}{N-D} \cdot \frac{N-D-n+x}{D+1-x} & \text{if } n-N+D+1 \le x \le D, \\ \infty & \text{if } x = D+1. \end{cases}$$

Hence $p_{D+1}(x)/p_D(x)$ is increasing in x. Thus, the HG distribution gives MLR.

(iii) [+2] Show an example of the continuous distribution with MLR (with proof) (except the normal with known variance).

Answer:

Consider the exponential distribution

$$f_{\lambda}(x) = \lambda e^{-\lambda x}, \quad x > 0, \quad \lambda > 0$$

and the hypothesis testing

$$H_0 : \lambda \leq \lambda_0 \qquad \text{versus} \qquad H_1 : \lambda > \lambda_0 .$$

For $\lambda' > \lambda$, the likelihood ratio

$$\frac{f_{\lambda'}(x)}{f_{\lambda}(x)} = \frac{\lambda' e^{-\lambda' x}}{\lambda e^{-\lambda x}} \propto e^{(\lambda' - \lambda)(-x)}$$

is increasing in $T(x) = -x$. Therefore, the exponential distribution is MLR in $T(x) = -x$.

(iv) [+2] Show an example of the discrete distribution with MLR (with proof) (except the binomial, HG, Poisson).

Answer:

Consider the geometric distribution

$$f_p(x) = \Pr_p(X = x) = p(1-p)^x, \quad x = 0, 1, 2, \cdots, \quad 0 < p < 1$$

and the hypothesis testing

$$H_0 : p \leq p_0 \qquad \text{versus} \qquad H_1 : p > p_0 .$$

For $p' > p$, the likelihood ratio

$$\frac{f_{p'}(x)}{f_p(x)} = \frac{p'(1-p')^x}{p(1-p)^x} \propto \left(\frac{1-p'}{1-p} \right)^x \propto \left(\frac{1-p}{1-p'} \right)^{-x}$$

is increasing in $T(x) = -x$ since $(1-p)/(1-p') > 1$. Therefore, the geometric distribution is MLR in $T(x) = -x$.