Homework#2 Statistical Inference III

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Example:

Consider two discrete distribution P_0 and P_1 with probability mass functions

$$p_0(x) = \begin{cases} 1/10 & \text{if } x = 0, \\ 1/10 & \text{if } x = 1, \\ 4/10 & \text{if } x = 2, \\ 4/10 & \text{if } x = 3. \end{cases} \quad \text{and} \quad p_1(x) = \begin{cases} 1/3 & \text{if } x = 0, \\ 1/6 & \text{if } x = 1, \\ 1/6 & \text{if } x = 2, \\ 1/3 & \text{if } x = 3. \end{cases}$$

respectively.

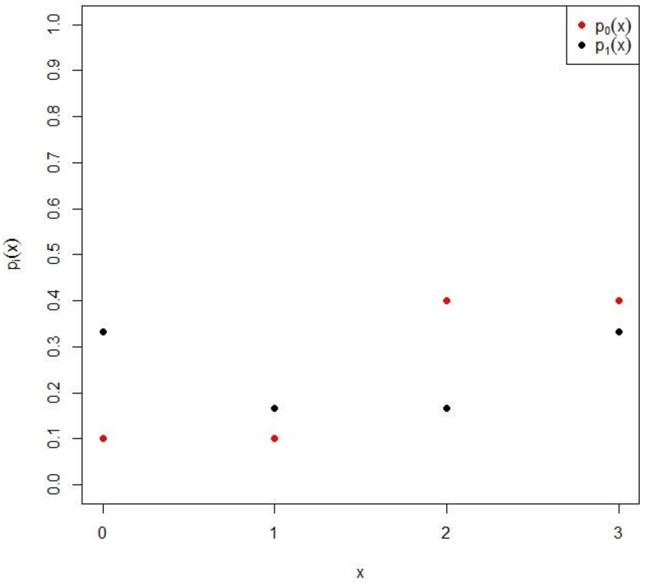

Figure 1. Probability mass functions $p_0(x)$ and $p_1(x)$.

Question:

Suppose $X \sim P = \{P_0, P_1\}$ and consider the simple hypothesis

$$H_0 : P = P_0 \quad \text{versus} \quad H_1 : P = P_1 .$$

Construct an UMP test under $\alpha = 0.3$.

Solution:

To find the rejection region S_1 of an UMP test under $\alpha = 0.3$, we first compute $r(x) = p_1(x)/p_0(x)$, $x = 0, 1, 2, 3$ (Table 1).

Table 1. The value of $r(x) = p_1(x)/p_0(x)$ with $x = 0, 1, 2, 3$.

x	0	1	2	3
$r(x)$	$\frac{10}{3}$	$\frac{5}{3}$	$\frac{5}{12}$	$\frac{5}{6}$

We select $x \in \{0, 1, 2, 3\}$ with the value $r(x)$ is large, that is

$$S_1 = \left\{ x_{(1)} = \arg\max_{x} r(x), x_{(2)} = \arg\max_{x \neq x_{(1)}} r(x), \cdots, x_{(k)} = \arg\max_{x \neq x_{(1)}, \cdots, x \neq x_{(k-1)}} r(x) \right\},$$

where $x_{(1)} = 0$, $x_{(2)} = 1$, $x_{(3)} = 3$ and $x_{(4)} = 2$. We choose k such that

$$\sum_{x \in S_1} p_0(x) \leq \alpha = 0.3.$$

Then the rejection region is

$$S_1 = \{ x : r(x) > c \},$$

where c is a constant such that

$$\sum_{x \in S_1} p_0(x) \leq 0.3.$$

Now, we can plot the figure of the Type I error function

$$\alpha(c) = \sum_{r(x) > c} p_0(x)$$

and the Power function

$$\beta(c) = \sum_{r(x) > c} p_0(x).$$

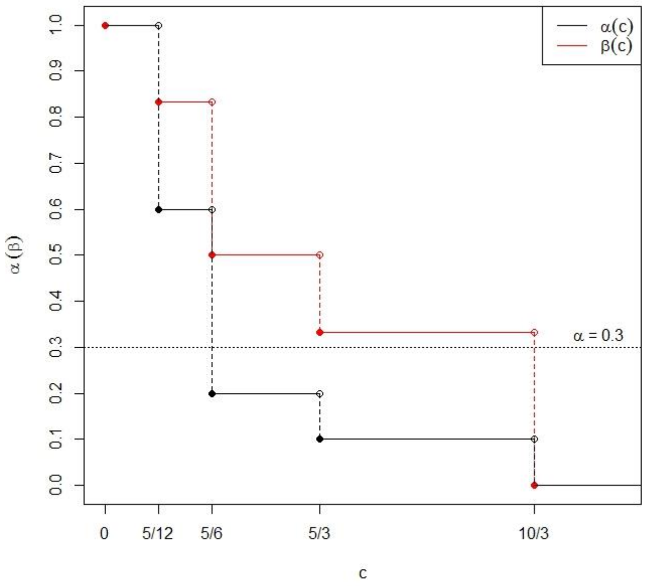

Figure 2. The Type I error function $\alpha(c)$ and the Power function $\beta(c)$.

To maximize power we choose c^* such that

$$\sum_{r(x)>c^*} p_0(x) = 0.3.$$

But Figure 2 reveals that there does not exist c^* satisfies the equation above. Therefore, we have to consider the critical function

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0,1, \\ \dfrac{\alpha - \alpha(c_0)}{\alpha(c_0^-) - \alpha(c_0)} & \text{if } x = 3, \\ 0 & \text{if } x = 2. \end{cases}$$

where $\alpha(c_0) = \alpha(5/6) = 2/10$ and $\alpha(c_0^-) = \alpha(5/12) = 6/10$. Thus, the critical function becomes

$$\phi(x) = \begin{cases} 1 & \text{if } x = 0,1, \\ 1/4 & \text{if } x = 3, \\ 0 & \text{if } x = 2. \end{cases}$$

We check the expectation of the critical function under the null hypothesis (Type I error)

$$
\begin{aligned}
E_0\{\phi(X)\} &= 1\times P_0(X=1 \text{ or } 2)+\frac{1}{4}\times P_0(X=3)\\
&= 1\times\left(\frac{1}{10}+\frac{1}{10}\right)+\frac{1}{4}\times\frac{4}{10}\\
&= 0.3\\
&= \alpha.
\end{aligned}
$$

The Type I error is exactly equal 0.3. Hence we have construct a UMP test under $\alpha = 0.3$. If $x = 1 \text{ or } 2$, we directly reject the null hypothesis and if $x = 3$, we reject the null hypothesis with probability equal 0.25.