Homework#6 Statistical Inference II

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Problem 6.2 [p.509]

In Example 6.1, verify Equation (6.4).

Solution:

In Example 6.1, we have $X_1, \cdots, X_n$ be i.i.d. according to a two-parameter Weibull distribution, whose density is

$$f(x \mid \beta, \gamma) = \frac{\gamma}{\beta} x^{\gamma-1} e^{-\frac{x^\gamma}{\beta}}, \qquad x > 0, \ \beta > 0, \ \gamma > 0.$$

The likelihood function is

$$L(\beta, \gamma) = \prod_{i=1}^n f(x_i \mid \beta, \gamma) = \left(\frac{\gamma}{\beta}\right)^n \prod_{i=1}^n x_i^{\gamma-1} e^{-\frac{1}{\beta}\sum_{i=1}^n x_i^\gamma}.$$

The log-likelihood function is

$$\ell(\beta, \gamma) = \log L(\beta, \gamma) = n \log \gamma - n \log \beta + (\gamma - 1) \sum_{i=1}^n \log x_i - \frac{1}{\beta} \sum_{i=1}^n x_i^\gamma.$$

Then we can derive the score functions:

$$\frac{\partial \ell(\beta, \gamma)}{\partial \beta} = -\frac{n}{\beta} + \frac{1}{\beta^2} \sum_{i=1}^n x_i^\gamma,$$
$$\frac{\partial \ell(\beta, \gamma)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^n \log x_i - \frac{1}{\beta} \sum_{i=1}^n x_i^\gamma \log x_i.$$

Setting the score functions equal to zero and solve β and γ, we have

$$-\frac{n}{\beta}+\frac{1}{\beta^2}\sum_{i=1}^{n}x_i^{\gamma}=0\Rightarrow\beta=\frac{1}{n}\sum_{i=1}^{n}x_i^{\gamma}.$$

Therefore, we obtain

$$\beta=\frac{1}{n}\sum_{i=1}^{n}x_i^{\gamma}.$$

Another equation is

$$\frac{n}{\gamma}+\sum_{i=1}^{n}\log x_i-\frac{1}{\beta}\sum_{i=1}^{n}x_i^{\gamma}\log x_i=0\Rightarrow\frac{1}{n\beta}\sum_{i=1}^{n}x_i^{\gamma}\log x_i=\frac{1}{\gamma}+\frac{1}{n}\sum_{i=1}^{n}\log x_i.$$

Here, we can replace β, the equation above can be rewrite as

$$h(\gamma)\equiv\frac{\sum_{i=1}^{n}x_i^{\gamma}\log x_i}{\sum_{i=1}^{n}x_i^{\gamma}}-\frac{1}{\gamma}=\frac{1}{n}\sum_{i=1}^{n}\log x_i.$$

To show that h function has at most one solution, we need to derive the first derivatives of h, that is

$$h'(\gamma)=\frac{\partial}{\partial\gamma}\left(\frac{\sum_{i=1}^{n}x_i^{\gamma}\log x_i}{\sum_{i=1}^{n}x_i^{\gamma}}-\frac{1}{\gamma}\right)=\frac{\sum_{i=1}^{n}x_i^{\gamma}(\log x_i)^2\sum_{i=1}^{n}x_i^{\gamma}-\left(\sum_{i=1}^{n}x_i^{\gamma}\log x_i\right)^2}{\left(\sum_{i=1}^{n}x_i^{\gamma}\right)^2}+\frac{1}{\gamma^2}$$

$$=\frac{\sum_{i=1}^{n}x_i^{\gamma}(\log x_i)^2}{\sum_{i=1}^{n}x_i^{\gamma}}-\left(\frac{\sum_{i=1}^{n}x_i^{\gamma}\log x_i}{\sum_{i=1}^{n}x_i^{\gamma}}\right)^2+\frac{1}{\gamma^2}$$

$$=\sum_{i=1}^{n}(\log x_i)^2\frac{x_i^{\gamma}}{\sum_{i=1}^{n}x_i^{\gamma}}-\left(\sum_{i=1}^{n}\log x_i\frac{x_i^{\gamma}}{\sum_{i=1}^{n}x_i^{\gamma}}\right)^2+\frac{1}{\gamma^2}.$$

Consider the following transformation,

$$a_i = \log x_i, \qquad p_i = \frac{x_i^{\gamma}}{\sum_{i=1}^{n} x_i^{\gamma}} = \frac{e^{\gamma a_i}}{\sum_{i=1}^{n} e^{\gamma a_i}} \quad \left(\sum_{i=1}^{n} p_i = 1 \right).$$

Then we have

$$h'(\gamma) = \sum_{i=1}^{n} a_i^2 p_i - \left(\sum_{i=1}^{n} a_i p_i \right)^2 + \frac{1}{\gamma^2}.$$

The first two term can be seen as the variance of a discrete random variable with probability mass function is

$$A \sim P(A = a_i) = p_i, \quad i = 1, 2, \cdots, n.$$

Therefore, we obtain

$$h'(\gamma) = \mathrm{var}(A) + \frac{1}{\gamma^2} > 0, \qquad \text{for all} \quad \gamma > 0.$$

Hence we have shown that h function is strictly increasing in γ and also verify Equation (6.4).

Problem 6.3 [p.509]

Verify (6.5).

Solution:

The h function is

$$h(\gamma)=\frac{\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma}}-\frac{1}{\gamma}.$$

Letting $\gamma\to 0$, we have

$$\lim_{\gamma\to 0} h(\gamma)=\frac{\lim_{\gamma\to 0}\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\lim_{\gamma\to 0}\sum_{i=1}^{n} x_i^{\gamma}}-\lim_{\gamma\to 0}\frac{1}{\gamma}=\frac{1}{n}\sum_{i=1}^{n}\log x_i-\infty=-\infty.$$

Letting $\gamma\to\infty$, we have

$$\begin{aligned}
\lim_{\gamma\to\infty} h(\gamma)&=\lim_{\gamma\to\infty}\left(\frac{\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma}}\right)-\lim_{\gamma\to\infty}\frac{1}{\gamma}=\lim_{\gamma\to\infty}\left(\frac{\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma}}\right)\\
&=\lim_{\gamma\to\infty}\left(\frac{x_1^{\gamma}\log x_1}{x_1^{\gamma}+x_2^{\gamma}+\cdots+x_n^{\gamma}}+\frac{x_2^{\gamma}\log x_2}{x_1^{\gamma}+x_2^{\gamma}+\cdots+x_n^{\gamma}}+\cdots+\frac{x_n^{\gamma}\log x_n}{x_1^{\gamma}+x_2^{\gamma}+\cdots+x_n^{\gamma}}\right)\\
&=\lim_{\gamma\to\infty}\left(\frac{x_{(1)}^{\gamma}\log x_{(1)}}{x_{(1)}^{\gamma}+x_{(2)}^{\gamma}+\cdots+x_{(n)}^{\gamma}}+\frac{x_{(2)}^{\gamma}\log x_{(2)}}{x_{(1)}^{\gamma}+x_{(2)}^{\gamma}+\cdots+x_{(n)}^{\gamma}}+\cdots+\frac{x_{(n)}^{\gamma}\log x_{(n)}}{x_{(1)}^{\gamma}+x_{(2)}^{\gamma}+\cdots+x_{(n)}^{\gamma}}\right)\\
&=\lim_{\gamma\to\infty}\frac{\log x_{(1)}}{1+\left(\frac{x_{(2)}}{x_{(1)}}\right)^{\gamma}+\cdots+\left(\frac{x_{(n)}}{x_{(1)}}\right)^{\gamma}}+\lim_{\gamma\to\infty}\frac{\log x_{(2)}}{\left(\frac{x_{(1)}}{x_{(2)}}\right)^{\gamma}+1+\cdots+\left(\frac{x_{(n)}}{x_{(2)}}\right)^{\gamma}}\\
&\qquad+\cdots+\lim_{\gamma\to\infty}\frac{\log x_{(n)}}{\left(\frac{x_{(1)}}{x_{(n)}}\right)^{\gamma}+\left(\frac{x_{(2)}}{x_{(n)}}\right)^{\gamma}+\cdots+1}\\
&=\log x_{(n)}.
\end{aligned}$$

Since the logarithmic transformation does not change the ordering, we let $\log x_i = y_i$ for $i = 1, 2, \cdots, n$. Then we have

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} y_{(i)} < ny_{(n)} \Rightarrow \frac{1}{n}\sum_{i=1}^{n} y_{(i)} < y_{(n)} \Rightarrow \frac{1}{n}\sum_{i=1}^{n} \log x_{(i)} < \log x_{(n)}$$

Thus, we obtain

$$\lim_{\gamma \to 0} h(\gamma) = -\infty < \frac{1}{n}\sum_{i=1}^{n} \log x_i < \log x_{(n)} = \lim_{\gamma \to \infty} h(\gamma).$$

Combing the result in the previous problem, that is, h function is strictly increasing in γ. Therefore, we have proven h function has at most one solution and also verify Equation (6.5).

Example 6.1

To demonstrate the conclusion of Example 6.1, we need to generate data from the Weibull distribution by inverse transform method. The cumulative distribution function of the Weibull distribution is

$$F(x|\beta,\gamma)=1-e^{-\frac{x^\gamma}{\beta}}, \qquad x>0,\ \beta>0,\ \gamma>0.$$

Therefore, we have

$$F(X|\beta,\gamma)=1-e^{-\frac{X^\gamma}{\beta}}=U \Rightarrow X=\{-\beta\log(1-U)\}^{\frac{1}{\gamma}},$$

where $U \sim U(0,1)$. Then we can perform inverse transform method. Now we generate 10,000 samples from the Weibull distribution with $(\beta,\gamma)=(5,4)$ and plot the h function. Figure 1 shows that $\lim_{\gamma\to\infty} h(\gamma)=\log x_{(n)}$, which are derived in problem 6.3 is correct and the conclusion of example 6.1 is verified.

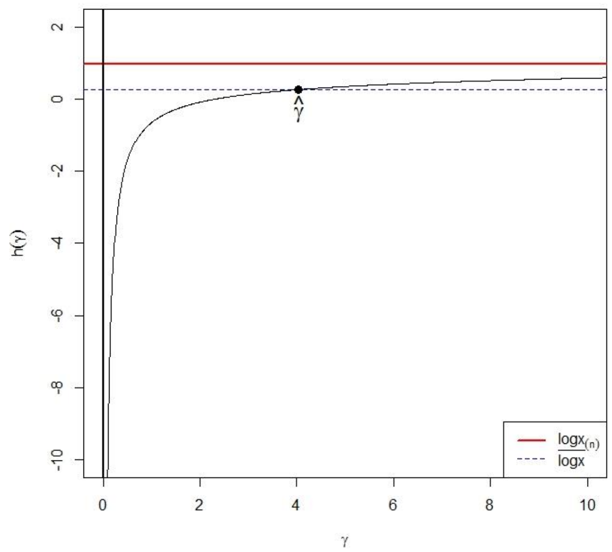

Fig. 1 The plot of $h(\gamma)=\sum_{i=1}^{n} x_i^\gamma \log x_i / \sum_{i=1}^{n} x_i^\gamma - 1/\gamma$.

To obtain the maximum likelihood estimator (MLE), we have to solve the score functions:

$$\frac{\partial \ell(\beta,\gamma)}{\partial \beta} = -\frac{n}{\beta} + \frac{1}{\beta^2}\sum_{i=1}^{n} x_i^{\gamma}, \qquad \frac{\partial \ell(\beta,\gamma)}{\partial \gamma} = \frac{n}{\gamma} + \sum_{i=1}^{n} \log x_i - \frac{1}{\beta}\sum_{i=1}^{n} x_i^{\gamma} \log x_i.$$

In problem 6.2, we have shown that the solution of score function equal to zero is the same as

$$\beta = \frac{1}{n}\sum_{i=1}^{n} x_i^{\gamma}, \qquad \frac{\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma}} - \frac{1}{\gamma} = \frac{1}{n}\sum_{i=1}^{n} \log x_i.$$

The second equation can be rewrite as

$$\gamma = \left(\frac{\sum_{i=1}^{n} x_i^{\gamma}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma}} - \frac{1}{n}\sum_{i=1}^{n} \log x_i \right)^{-1}.$$

Then we can perform the fixed-point iteration method to solve for $\hat{\gamma}$. The algorithm are given as follows:

Algorithm: The fixed-point iteration method

Step1. Repeat the fixed-point iterations

$$\gamma^{(k+1)} = \left(\frac{\sum_{i=1}^{n} x_i^{\gamma^{(k)}}\log x_i}{\sum_{i=1}^{n} x_i^{\gamma^{(k)}}} - \frac{1}{n}\sum_{i=1}^{n} \log x_i \right)^{-1}.$$

Step2. If $|\gamma^{(k+1)} - \gamma^{(k)}| < 10^{-5}$ then stop, the MLE is $\hat{\gamma} = \gamma^{(k+1)}$.

Once we obtain $\hat{\gamma}$, then the MLE of β is

$$\hat{\beta} = \frac{1}{n}\sum_{i=1}^{n} x_i^{\hat{\gamma}}.$$

The results are given in Table 1. Table 1 reveals that with sample size increase, the MLEs are more close to the true parameters. Figure 1 also conforms that

$$h(\hat{\gamma}) = \frac{1}{n}\sum_{i=1}^{n} \log x_i .$$

True parameters	n	$\hat{\beta}$	$\hat{\gamma}$
$\beta = 5$	50	5.569512	3.996342
$\gamma = 4$	500	6.082812	4.356211
	5000	5.138241	4.065424
	10000	5.076605	4.038097

Table 1. The MLEs of the Weibull distribution

Problem 5.26 [p.68]

If (X, Y) is distributed according to the bivariate normal distribution

$$\frac{1}{2\pi\sigma\tau\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left\{\left(\frac{x-\xi}{\sigma}\right)^2-2\rho\left(\frac{x-\xi}{\sigma}\right)\left(\frac{y-\eta}{\tau}\right)+\left(\frac{y-\eta}{\tau}\right)^2\right\}\right]$$

with $\xi=\eta=0$:

(a) Show that the moment generating function of (X, Y) is

$$M_{X,Y}(u_1,u_2)=e^{(u_1^2\sigma^2+2\rho\sigma\tau u_1u_2+u_2^2\tau^2)/2}.$$

Solution:

By straightforward calculation, we have

$$\begin{aligned}
M_{X,Y}(u_1,u_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{u_1x+u_2y}\frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma^2}-2\rho\frac{xy}{\sigma\tau}+\frac{y^2}{\tau^2}\right)\right\}dxdy \\
&= \frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}}\int_{-\infty}^{\infty}\exp\left\{u_2y-\frac{y^2}{2(1-\rho^2)\tau^2}\right\} \\
&\quad\times\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma^2}-2\rho\frac{xy}{\sigma\tau}\right)+u_1x\right\}dxdy.
\end{aligned}$$

Consider the integral

$$\begin{aligned}
&\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2(1-\rho^2)}\left(\frac{x^2}{\sigma^2}-2\rho\frac{xy}{\sigma\tau}\right)+u_1x\right\}dx \\
&= \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}\left\{x^2-2\left(\frac{\rho y\sigma}{\tau}+(1-\rho^2)\sigma^2u_1\right)x\right\}\right]dx \\
&= \int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma^2(1-\rho^2)}\left\{x-\left(\frac{\rho y\sigma}{\tau}+(1-\rho^2)\sigma^2u_1\right)\right\}^2\right]dx \\
&\qquad\times\exp\left\{\frac{\rho^2y^2}{2\tau^2(1-\rho^2)}+\frac{(1-\rho^2)\sigma^2u_1^2}{2}+\frac{\rho y\sigma u_1}{\tau}\right\} \\
&= \sqrt{2\pi(1-\rho^2)\sigma^2}\exp\left\{\frac{\rho^2y^2}{2\tau^2(1-\rho^2)}+\frac{(1-\rho^2)\sigma^2u_1^2}{2}+\frac{\rho y\sigma u_1}{\tau}\right\}.
\end{aligned}$$

Therefore, we have

$$
\begin{aligned}
M_{X,Y}(u_1,u_2) &= \frac{1}{\sqrt{2\pi}\tau}\exp\left\{\frac{(1-\rho^2)\sigma^2u_1^2}{2}\right\}\int_{-\infty}^{\infty}\exp\left\{-\frac{y^2}{2\tau^2}+\left(\frac{\rho\sigma u_1}{\tau}+u_2\right)y\right\}dy \\
&= \frac{1}{\sqrt{2\pi}\tau}\exp\left\{\frac{(1-\rho^2)\sigma^2u_1^2}{2}\right\}\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\tau^2}\{y^2-2(\rho\sigma\tau u_1+u_2\tau^2)y\}\right]dy \\
&= \frac{1}{\sqrt{2\pi}\tau}\exp\left\{\frac{(1-\rho^2)\sigma^2u_1^2}{2}\right\}\exp\left\{\frac{\rho^2\sigma^2u_1^2}{2}+\frac{u_2^2\tau^2}{2}+\rho\sigma\tau u_1u_2\right\} \\
&\times\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\tau^2}\{y-(\rho\sigma\tau u_1+u_2\tau^2)\}^2\right]dy \\
&= e^{(u_1^2\sigma^2+2\rho\sigma\tau u_1u_2+u_2^2\tau^2)/2}.
\end{aligned}
$$

Hence we have shown that

$$M_{X,Y}(u_1,u_2) = e^{(u_1^2\sigma^2+2\rho\sigma\tau u_1u_2+u_2^2\tau^2)/2}.$$

(b) Use (a) to show that

$$\mu_{12} = \mu_{21} = 0, \quad \mu_{11} = \rho\sigma\tau,$$

$$\mu_{13} = 3\rho\sigma\tau^3, \quad \mu_{31} = 3\rho\sigma^3\tau, \quad \mu_{22} = (1-\rho^2)\sigma^2\tau^2.$$

Solution:

By (a), we have

$$\begin{aligned}\frac{\partial^2}{\partial u_1 \partial u_2} M_{X,Y}(u_1, u_2) &= \frac{\partial}{\partial u_2}\left[(\sigma^2 u_1 + \rho\sigma\tau u_2)\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}\right] \\ &= (\rho\sigma\tau + \sigma^2\tau^2 u_1 u_2 + \rho\sigma^3\tau u_1^2 + \rho\sigma\tau^3 u_2^2 + \rho^2\sigma^2\tau^2 u_1 u_2) \\ &\quad \times \exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}.\end{aligned}$$

Then we obtain

$$\mu_{11} = \frac{\partial^2}{\partial u_1 \partial u_2} M_{X,Y}(u_1, u_2)\Big|_{u_1=u_2=0} = \rho\sigma\tau.$$

$$\begin{aligned}\frac{\partial^3}{\partial u_1 \partial u_2^2} M_{X,Y}(u_1, u_2) &= \frac{\partial}{\partial u_2}\Big[(\rho\sigma\tau + \sigma^2\tau^2 u_1 u_2 + \rho\sigma^3\tau u_1^2 + \rho\sigma\tau^3 u_2^2 + \rho^2\sigma^2\tau^2 u_1 u_2) \\ &\quad \times \exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}\Big] \\ &= (\sigma^2\tau^2 u_1 + 2\rho\sigma\tau^3 u_2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma\tau^3 u_2 + \sigma^2\tau^4 u_1 u_2^2 \\ &\quad + \rho\sigma^3\tau^3 u_1^2 u_2 + \rho\sigma\tau^5 u_2^3 + \rho^2\sigma^2\tau^4 u_1 u_2^2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma^3\tau^3 u_1^2 u_2 \\ &\quad + \rho^2\sigma^4\tau^2 u_1^3 + \rho^2\sigma^2\tau^4 u_1 u_2^3)\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}\Big].\end{aligned}$$

Then we obtain

$$\mu_{12} = \frac{\partial^3}{\partial u_1 \partial u_2^2} M_{X,Y}(u_1, u_2)\Big|_{u_1=u_2=0} = 0.$$

Since μ_{21} is equal to μ_{12} by exchanging σ and τ, thus, we have $\mu_{21} = 0$.

One can notice that we only need to consider the term without u_1 and u_2, therefore,

$$\begin{aligned}
&\frac{\partial^4}{\partial u_1 \partial u_2^3} M_{X,Y}(u_1,u_2) \\
&= \frac{\partial}{\partial u_2}\Big[(\sigma^2\tau^2 u_1 + 2\rho\sigma\tau^3 u_2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma\tau^3 u_2 + \sigma^2\tau^4 u_1 u_2^2 \\
&+ \rho\sigma^3\tau^3 u_1^2 u_2 + \rho\sigma\tau^5 u_2^3 + \rho^2\sigma^2\tau^4 u_1 u_2^2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma^3\tau^3 u_1^2 u_2 \\
&+ \rho^2\sigma^4\tau^2 u_1^3 + \rho^2\sigma^2\tau^4 u_1 u_2^3)\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}\Big] \\
&= \{3\rho\sigma\tau^3 + K_1(u_1,u_2)\}\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\},
\end{aligned}$$

where $K_1(0,0)=0$. Then we obtain

$$\mu_{13} = \left.\frac{\partial^4}{\partial u_1 \partial u_2^3} M_{X,Y}(u_1,u_2)\right|_{u_1=u_2=0} = 3\rho\sigma\tau^3.$$

Since μ_{31} is equal to μ_{13} by exchanging σ and τ, thus, we have $\mu_{31} = 3\rho\sigma^3\tau.$

$$\begin{aligned}
&\frac{\partial^4}{\partial u_1^2 \partial u_2^2} M_{X,Y}(u_1,u_2) \\
&= \frac{\partial}{\partial u_1}\Big[(\sigma^2\tau^2 u_1 + 2\rho\sigma\tau^3 u_2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma\tau^3 u_2 + \sigma^2\tau^4 u_1 u_2^2 \\
&+ \rho\sigma^3\tau^3 u_1^2 u_2 + \rho\sigma\tau^5 u_2^3 + \rho^2\sigma^2\tau^4 u_1 u_2^2 + \rho^2\sigma^2\tau^2 u_1 + \rho\sigma^3\tau^3 u_1^2 u_2 \\
&+ \rho^2\sigma^4\tau^2 u_1^3 + \rho^2\sigma^2\tau^4 u_1 u_2^3)\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\}\Big] \\
&= \{\sigma^2\tau^2 + 2\rho^2\sigma^2\tau^2 + K_2(u_1,u_2)\}\exp\left\{\frac{1}{2}(u_1^2\sigma^2 + 2\rho\sigma\tau u_1 u_2 + u_2^2\tau^2)\right\},
\end{aligned}$$

where $K_2(0,0)=0$. Then we obtain

$$\mu_{22} = \left.\frac{\partial^4}{\partial u_1^2 \partial u_2^2} M_{X,Y}(u_1,u_2)\right|_{u_1=u_2=0} = (1+2\rho^2)\sigma^2\tau^2.$$

Appendix

Algorithm: Data generation of Weibull distribution

Step1: Generate $U_i \sim U(0,1)$, for $i = 1, 2, \cdots, n$.

Step2: Set $X_i = \{-\beta \log U_i\}^{1/\gamma}$, for $i = 1, 2, \cdots, n$.

R code

```
### parameter ###

n       = 10000
gamma = 4
beta   = 5

### generate weibull data ###

set.seed(615)
u=runif(n,0,1)
x=(-beta*log(u))^(1/gamma)

### recursive iteration to solve MLE ###

count=0
gamma_old=mean(x)
repeat {

   r1=sum(x^gamma_old*log(x))/sum(x^gamma_old)
   r2=mean(log(x))
   gamma_new=1/(r1-r2)

   if (abs(gamma_new-gamma_old)<10^-5) {break}
   gamma_old=gamma_new
   count=count+1

}
count
```

```
gamma_hat=gamma_new; gamma_hat; gamma
beta_hat=mean(x^gamma_hat); beta_hat ;beta

### h function ###
h_func=function(gamma) {

  sum(x^gamma*log(x))/sum(x^gamma)-1/gamma

}

### plot h function ###
q=seq(0.01,100,0.1)
plot(q,h_func(q),type="l",ylim=c(-10,2),xlim=c(0,10),ylab=expression(h(gamma)),xlab
=expression(gamma))
legend("bottomright",c(expression(logx[(n)]),expression(bar(logx))),col=c("red","blue")
,lwd=c(2,1),lty=c(1,2))

h_func(700); max(log(x))
abline(h=max(log(x)),lwd=2,col="red")

h_func(10^-5)
abline(v=0,lwd=2)

mean(log(x))
abline(h=mean(log(x)),col="blue",lty=2)
```