Homework#3 Statistical Inference II

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Problem 1.1 [p.389]

For the situation of Example 1.2:

$$X \sim \text{Bin}(n, p), \qquad p \sim \text{Beta}(a, b).$$

Then the Bayes estimator is

$$\delta = \frac{a+x}{a+b+n}.$$

Consider a group of three estimators $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$, Bayes estimator of p from $\text{Beta}(1,3)$, $\text{Beta}(2,2)$ and $\text{Beta}(3,1)$ priors, respectively.

(a) Plot the risk functions of $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$ for $n = 5, 10, 25$.

Solution:

The risk function of δ is

$$\begin{aligned} R(p, \delta) &= E\{(p-\delta)^2\} = E\{(\delta - p)^2\} = E\left\{\left(\frac{a+x}{a+b+n} - p\right)^2\right\} \\ &= \frac{1}{(a+b+n)^2} E[\{(a+x) - (a+b+n)p\}^2] \\ &= \frac{1}{(a+b+n)^2} E([(x-np) + \{a(1-p) - bp\}]^2) \\ &= \frac{1}{(a+b+n)^2} [E\{(x-np)^2\} + \{a(1-p) - bp\}^2] \\ &= \frac{1}{(a+b+n)^2} [np(1-p) + \{a(1-p) - bp\}^2]. \end{aligned}$$

Therefore, the risk function of $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$ are

$$R(p,\delta^{1/4})=\frac{1}{(4+n)^2}[\,np(1-p)+\{(1-p)-3p\}^2\,],$$

$$R(p,\delta^{1/2})=\frac{1}{(4+n)^2}[\,np(1-p)+\{2(1-p)-2p\}^2\,],$$

$$R(p,\delta^{3/4})=\frac{1}{(4+n)^2}[\,np(1-p)+\{3(1-p)-p\}^2\,],$$

respectively. Now, we can use R software to plot the risk functions of $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$ for $n=5,10,25$. The results are compared in Figure 1.

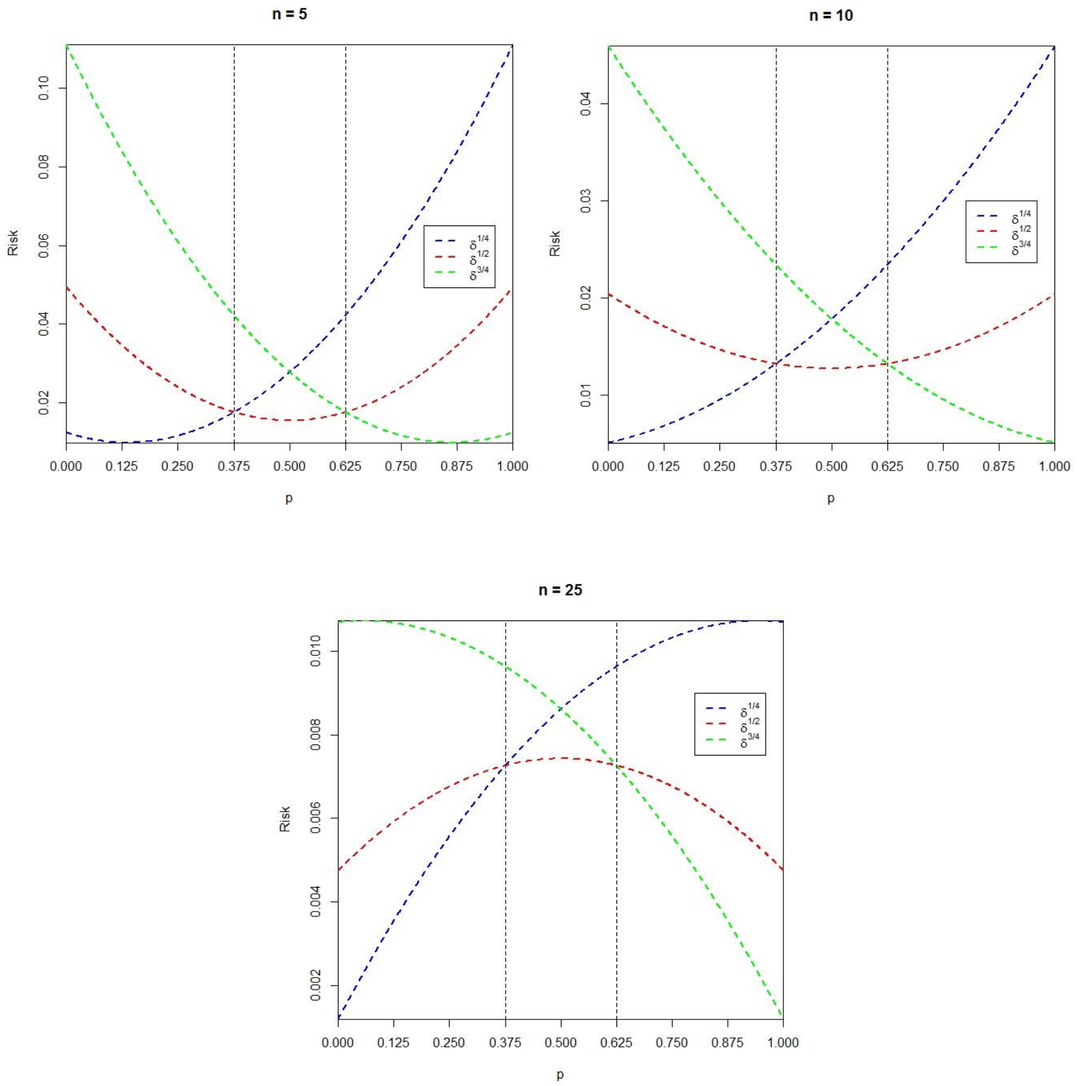

Fig. 1. Risk functions of $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$ for $n=5,10,25$.

(b) For each value of n in part (a), find the range of prior values of p for which each estimator is preferred.

Solution:

For the case $n=5$, the cross points in Figure 1 are solved by

$$
\begin{aligned}
&R(p,\delta^{1/4}) = R(p,\delta^{1/2})\\
&\Rightarrow \frac{1}{9^2}[\,5p(1-p)+\{(1-p)-3p\}^2\,] = \frac{1}{9^2}[\,5p(1-p)+\{2(1-p)-2p\}^2\,]\\
&\Rightarrow 5p(1-p)+\{(1-p)-3p\}^2 = 5p(1-p)+\{2(1-p)-2p\}^2\\
&\Rightarrow \{(1-p)-3p\}^2 = \{2(1-p)-2p\}^2\\
&\Rightarrow \{1-4p\}^2 = \{2-4p\}^2 \Rightarrow 16p^2-8p+1 = 16p^2-16p+4\\
&\Rightarrow 8p=3 \Rightarrow p=3/8=0.375
\end{aligned}
$$

and

$$
\begin{aligned}
&R(p,\delta^{1/2}) = R(p,\delta^{3/4})\\
&\Rightarrow \frac{1}{9^2}[\,5p(1-p)+\{2(1-p)-2p\}^2\,] = \frac{1}{9^2}[\,5p(1-p)+\{3(1-p)-p\}^2\,]\\
&\Rightarrow 5p(1-p)+\{2(1-p)-2p\}^2 = 5p(1-p)+\{3(1-p)-p\}^2\\
&\Rightarrow \{2(1-p)-2p\}^2 = \{3(1-p)-p\}^2\\
&\Rightarrow \{2-4p\}^2 = \{3-4p\}^2 \Rightarrow 16p^2-16p+4 = 16p^2-24p+9\\
&\Rightarrow 8p=5 \Rightarrow p=5/8=0.625.
\end{aligned}
$$

The cross points of $n=10, 25$ in Figure 1 can also be solved similarly and they are all the same as the case $n=5$. Because we want the risk as small as possible. Therefore, for $n=5, 10, 25$, Figure 1 reveals the following conclusions:

1) Estimator $\delta^{1/4}$ is better with the range $0<p<0.375$.

2) Estimator $\delta^{1/2}$ is better with the range $0.375<p<0.625$.

3) Estimator $\delta^{3/4}$ is better with the range $0.625<p<1$.

(c) If an experimenter has no prior knowledge of p, which of $\delta^{1/4}$, $\delta^{1/2}$ and $\delta^{3/4}$ would you recommend? Justify your choice.

Solution:

If one has no prior knowledge of p, I would recommend the estimator with the curve of risk function is flat, that is I recommend $\delta^{1/2}$. More specifically, this "flat" means that the maximum risk is the smallest among the others. If the risk function is flat, this means the value of p doesn't effect the risk too much. Because if we don't have any prior knowledge, we don't know the distribution of p. Therefore, choosing an estimator which the value of p doesn't effect the risk too much is better.

R code

```
risk_func=function(a,b,n,p) {

  ((n*p*(1-p))+(a*(1-p)-b*p)^2)/(a+b+n)^2

}

n = 5
p = seq(0,1,0.001)

windows()
plot(p,risk_func(1,3,n,p),type="l",main="n = 5",col="blue",lty=2,lwd=2,
  xaxt="n",xaxs="i",yaxs="i",ylab="Risk")
axis(1,at=c(0,0.125,0.25,3/8,0.5,5/8,0.75,0.875,1))
legend(0.8,0.065,c(expression(delta^"1/4",delta^"1/2",delta^"3/4")),lty=2,lwd=2,col=c("blue","red","green"))
lines(p,risk_func(2,2,n,p),type="l",col="red",lty=2,lwd=2)
lines(p,risk_func(3,1,n,p),type="l",col="green",lty=2,lwd=2)
abline(v=3/8,lty=2)
abline(v=5/8,lty=2)

n = 10
p = seq(0,1,0.001)

windows()
plot(p,risk_func(1,3,n,p),type="l",main="n = 10",col="blue",lty=2,lwd=2,
  xaxt="n",xaxs="i",yaxs="i",ylab="Risk")
axis(1,at=c(0,0.125,0.25,3/8,0.5,5/8,0.75,0.875,1))
legend(0.8,0.03,c(expression(delta^"1/4",delta^"1/2",delta^"3/4")),lty=2,lwd=2,col=c("blue","red","green"))
lines(p,risk_func(2,2,n,p),type="l",col="red",lty=2,lwd=2)
lines(p,risk_func(3,1,n,p),type="l",col="green",lty=2,lwd=2)
abline(v=3/8,lty=2)
abline(v=5/8,lty=2)
```

```
n = 25
p = seq(0,1,0.001)

windows()
plot(p,risk_func(1,3,n,p),type="l",main="n = 25",col="blue",lty=2,lwd=2,
  xaxt="n",xaxs="i",yaxs="i",ylab="Risk")
axis(1,at=c(0,0.125,0.25,3/8,0.5,5/8,0.75,0.875,1))
legend(0.8,0.009,c(expression(delta^"1/4",delta^"1/2",delta^"3/4")),lty=2,lwd=2,col=c("
blue","red","green"))
lines(p,risk_func(2,2,n,p),type="l",col="red",lty=2,lwd=2)
lines(p,risk_func(3,1,n,p),type="l",col="green",lty=2,lwd=2)
abline(v=3/8,lty=2)
abline(v=5/8,lty=2)
```