Homework#5, Statistical Inference II, 2013 Spring

1. Let $T_1, T_2, \cdots, T_n$ be iid random variables from the density $f(x) = \lambda e^{-\lambda x} I(x \ge 0)$.

For a fixed number $r < n$, let $T_{(1)} < T_{(2)} < \cdots < T_{(r)}$ be the ordered samples for the r smallest observations (Type II censored data). Consider inference for $\lambda > 0$ based only on the observations $(T_{(1)}, T_{(2)}, \cdots, T_{(r)})$.

1) Write the joint density of $(T_{(1)}, T_{(2)}, \cdots, T_{(r)})$.

2) Show that $V = \sum_{i=1}^{r} T_{(i)} + (n-r)T_{(r)}$ is complete and sufficient for λ.

3) Find the distribution of $2\lambda V$, which is a pivotal quantity.

4) Find an exact $(1-\alpha)$-confidence sets for λ using the pivotal quantity.

2. In the least square model $X = Z\beta + \varepsilon$ with p =2, one has data

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> X
 [1,] -0.8
 [2,] -2.1
 [3,]  1.4
 [4,] -1.0
 [5,]  2.3
 [6,]  3.6
 [7,] -0.6
 [8,]  1.5
 [9,] -1.0
[10,]  2.2
> Z
      [,1] [,2]
 [1,] -0.6 -0.5
 [2,] -1.5  0.1
 [3,]  0.7 -0.1
 [4,]  0.1  0.8
 [5,]  0.7  0.3
 [6,]  1.1  1.5
 [7,]  0.6 -1.6
 [8,]  1.0  1.0
 [9,] -0.5  0.5
[10,]  0.4  1.0
```

Draw 95% confidence set for $\beta = (\beta_1, \beta_2)$

(detailed numerical information about your picture is necessary)

Answer 1

1)

$$\begin{aligned}
&\Pr(T_{(1)}=t_{(1)}, T_{(2)}=t_{(2)}, \cdots, T_{(r)}=t_{(r)}) \\
&=\binom{n}{r}\Pr(T_1=t_{(1)}, T_2=t_{(2)}, \cdots, T_r=t_{(r)}, T_{r+1}>t_{(r)}, \cdots, T_n>t_{(r)}) \\
&=\binom{n}{r}\left(\prod_{i=1}^{r}\lambda e^{-\lambda t_{(i)}}\right)\times e^{-\lambda(n-r)t_{(r)}}=\binom{n}{r}\lambda^r e^{-\lambda\left\{\sum_{i=1}^{r}t_{(i)}+(n-r)t_{(r)}\right\}}=\binom{n}{r}e^{-\lambda v+r\log\lambda}
\end{aligned}$$

2) Since the above density is in an exponential family with full rank, $V=\sum_{i=1}^{r}T_{(i)}+(n-r)T_{(r)}$ is complete and sufficient (Prop 2.1, p.110 of Shao).

3) Let $T_{(0)}=0$. Then,

$$\begin{aligned}
&T_{(1)}=T_{(1)}-T_{(0)} \\
&T_{(2)}=(T_{(1)}-T_{(0)})+(T_{(2)}-T_{(1)}) \\
&\vdots \\
&T_{(r)}=(T_{(1)}-T_{(0)})+(T_{(2)}-T_{(1)})+\cdots+(T_{(r)}-T_{(r-1)})
\end{aligned}$$

So,

$$\begin{aligned}
V&=\sum_{i=1}^{r}T_{(i)}+(n-r)T_{(r)}=\sum_{i=1}^{r}(r-i+1)(T_{(i)}-T_{(i-1)})+(n-r)\sum_{i=1}^{r}(T_{(i)}-T_{(i-1)}) \\
&=\sum_{i=1}^{r}(n-i+1)(T_{(i)}-T_{(i-1)})\equiv\sum_{i=1}^{r}U_i.
\end{aligned}$$

where $U_i\equiv(n-i+1)(T_{(i)}-T_{(i-1)}), i=1,\ldots,r$ follows iid with the pdf $f(x)=\lambda e^{-\lambda x}I(x\ge 0)$. Thus, $2\lambda U_i$ has the $f(x)=1/2e^{-x/2}I(x\ge 0)$, the chi-squared distribution with df=2. Hence, $V\sim\chi^2_{df=2r}$.

4) Find c_1 and c_2 such that

$\Pr(c_1\le\chi^2_{df=2r}\le c_2)=1-\alpha$.

Then, solve $c_1\le 2\lambda V\le c_2$ and get $\frac{c_1}{2V}\le\lambda\le\frac{c_2}{2V}$.