Homework#3, Statistical Inference II, 2013 Spring

Note that the likelihood ratio $\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$ is well-defined when at least one of $f_{\theta_1}(x)$ and $f_{\theta_2}(x)$ is nonzero [Def. 6.2 in the textbook]. Hence, we define the set $S=\{x: f_{\theta_1}(x)>0 \text{ or } f_{\theta_2}(x)>0\}$ in which $\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)}$ is well-defined.

1. Consider the family $\{f_\theta(x):\theta\in R\}$ with $f_\theta(x)=c(\theta)h(x)I_{(a(\theta),b(\theta))}(x)$, where $a(\theta)$ and $b(\theta)$ are nondecreasing [Ex. 12 in the textbook].

a) Define the set S.

b) Show that the family is MLR.

c) Derive a size α UMP test for $H_0:\theta\le\theta_0$ vs. $H_1:\theta>\theta_0$.

2. Data $(X_1,...,X_n)$ follows independently and identically a Weibull distribution with $P_\theta(X_i>x)=\exp\{-(x/\theta)^c\}I_{(0,\infty)}(x)+I_{(-\infty,0]}(x)$, where $\theta>0$ is unknown and $c>0$ is known. Derive a size α UMP test for $H_0:\theta\le\theta_0$ vs. $H_1:\theta>\theta_0$ [the cut-off value should use the percentile of the chi-square distribution].

This is a simplified answer. You need to write more details.

1.Answer

Case (i): $a(\theta_2) < b(\theta_1)$

$S = (a(\theta_1), b(\theta_2))$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \begin{cases} 0 & \text{if} \quad a(\theta_1) < x \le a(\theta_2) \\ c(\theta_2)/c(\theta_1) & \text{if} \quad a(\theta_2) < x < b(\theta_1) \\ \infty & \text{if} \quad b(\theta_1) \le x < b(\theta_2) \end{cases} \qquad \text{(MLR in x)}$$

Case (i): $b(\theta_1) \le a(\theta_2)$

$S = (a(\theta_1), b(\theta_1)) \cup (a(\theta_2), b(\theta_2))$

$$\frac{f_{\theta_2}(x)}{f_{\theta_1}(x)} = \begin{cases} 0 & \text{if} \quad a(\theta_1) < x \le b(\theta_1) \\ \infty & \text{if} \quad a(\theta_2) < x < b(\theta_2) \end{cases} \qquad \text{(MLR in x)}$$

c) $T(X) = I_{(c_\alpha, \infty)}(X)$, where $c_\alpha = H^{-1}\{H(b(\theta_0)) - \alpha / c(\theta_0)\}$

Answer 2:

Since $f_\theta(x_i) = c x_i^{c-1} I_{(0,\infty)}(x) \exp(-x_i^c / \theta^c - c \log \theta)$, the p.d.f. of full data is

$$f_\theta(x) = c^n \prod_{i=1}^{n} x_i^{c-1} I_{(0,\infty)}(x_{(1)}) \exp(-\sum_{i=1}^{n} x_i^c / \theta^c - nc \log \theta),$$

where $\eta(\theta) = -1/\theta^c$ is strictly increasing and $Y(X) = \sum_{i=1}^{n} X_i^c \sim Gamma(n, \theta^c)$.

Note that $2Y(X)/\theta^c \sim \chi^2_{df=n}$. Solving $P_{\theta_0}(Y > d) = P_{\theta_0}(\frac{2Y}{\theta_0^c} > \frac{2d}{\theta_0^c}) = \alpha$,

$d = \theta_0^c \chi^2_{df=n,1-\alpha} / 2$. Hence, by Corollary 6.2, the UMP is

$T(X) = I_{(d_\alpha, \infty)}(Y(X))$, where $d_\alpha = \theta_0^c \chi^2_{df=n,1-\alpha} / 2$.