Report#4 Statistical Inference I

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Problem 5.28

(a)

The distribution of X with density

$$p_\theta(x)=\exp\left[\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x)-B'(\theta)\right]h'(x).$$

The distribution of X truncated on A that is the distribution with density

$$\frac{p_\theta(x)I_A(x)}{P_\theta(A)}.$$

Therefore,

$$\begin{aligned}&\int_{\mathcal{X}}\frac{p_\theta(x)I_A(x)}{P_\theta(A)}dx=1\\ &\Rightarrow\int_A p_\theta(x)dx=P_\theta(A)\\ &\Rightarrow P_\theta(A)=\int_A\exp\left[\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x)-B'(\theta)\right]h'(x)dx.\end{aligned}$$

Then,

$$\begin{aligned}\frac{p_\theta(x)I_A(x)}{P_\theta(A)}&=\exp\left[\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x)-B'(\theta)\right]h'(x)\frac{I_A(x)}{P_\theta(A)}\\ &=\exp\left[\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x)-\{B'(\theta)+\log P_\theta(A)\}\right]h'(x)I_A(x)\\ &=\exp\left[\sum_{i=1}^{s}\eta_i(\theta)T_i(x)-B(\theta)\right]h(x).\end{aligned}$$

Hence the distribution of X truncated on A is an s-dimensional exponential family where

$$\eta_i(\theta)=\eta_i'(\theta),\ \ T_i(\theta)=T_i'(\theta),\ \ h(x)=h'(x)I_A(x)$$

and $$B(\theta)=B'(\theta)+\int_A\left\{\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x)-B'(\theta)+\log h'(x)\right\}dx.$$

Problem 6.37

Under the assumptions of Theorem 6.5, let A be any fixed set in the sample space, P_θ^* the distribution P_θ truncated on A and $\mathrm{P}^* = \{ P_\theta^*, \theta \in \Omega \}$. Then prove

(a) if T is sufficient for P, it is sufficient for P^*.

Proof:

Since T is sufficient for P, by the factorization criterion

$$p_\theta(x) = g_\theta[T(x)]h(x).$$

Therefore,

$$\begin{aligned} p_\theta^*(x) &= \frac{p_\theta(x) I_A(x)}{P_\theta(A)} \\ &= \frac{g_\theta[T(x)]h(x)I_A(x)}{P_\theta(A)}. \end{aligned}$$

Let

$$g_\theta^*(x) = \frac{g_\theta(x)}{P_\theta(A)}, \quad h^*(x) = h(x)I_A(x).$$

By the factorization criterion

$$\begin{aligned} p_\theta^*(x) &= \frac{g_\theta[T(x)]h(x)I_A(x)}{P_\theta(A)} \\ &= g_\theta^*[T(x)]h^*(x) \end{aligned}$$

Hence T is sufficient for P^*.

(b) if, in addition, if T is complete for P, it is complete for P^*

Proof:

Since T is complete for $\mathrm{P}=\{P_\theta,\theta\in\Omega\}$, we have

$$E_\theta[f(T(X))]=\int_\chi f(T(x))p_\theta(x)dx=0, \forall\theta\in\Omega \Rightarrow f=0.$$

Consider P_θ^* the distribution P_θ truncated on A and $\mathrm{P}^*=\{P_\theta^*,\theta\in\Omega\}$.

$$\begin{aligned}
&E_\theta[f(T(X))]=0\\
&\Rightarrow \int_\chi f(T(x))p_\theta^*(x)dx=0\\
&\Rightarrow \int_\chi f(T(x))\frac{p_\theta(x)I_A(x)}{P_\theta(A)}dx=0\\
&\Rightarrow \int_A f(T(x))p_\theta(x)dx=0.
\end{aligned}$$

Since A is any fixed set in the sample space. Therefore,

$$\int_A f(T(x))p_\theta(x)dx=0, \forall\theta\in\Omega \Rightarrow f=0.$$

Hence T is complete for P^*.

Problem 7.4

If ϕ is convex on (a,b) and ψ is convex and non-decreasing on the range of ϕ, show that the function $\psi[\phi(x)]$ is convex on (a,b).

Proof:

Since ϕ is convex on (a,b), for all $\gamma \in (0,1)$ and for all $x, y \in (a,b)$ we have

$$\phi(\gamma x + (1-\gamma)y) \le \gamma\phi(x) + (1-\gamma)\phi(y).$$

Since ψ is non-decreasing on the range of ϕ, we have

$$\psi\{\phi(\gamma x + (1-\gamma)y)\} \le \psi\{\gamma\phi(x) + (1-\gamma)\phi(y)\}$$

and ψ is also convex on the range of ϕ, hence

$$\psi\{\gamma\phi(x) + (1-\gamma)\phi(y)\} \le \gamma\psi\{\phi(x)\} + (1-\gamma)\psi\{\phi(y)\}.$$

Therefore, we obtain

$$\psi\{\phi(\gamma x + (1-\gamma)y)\} \le \gamma\psi\{\phi(x)\} + (1-\gamma)\psi\{\phi(y)\},$$

for all $\gamma \in (0,1)$ and for all $x, y \in (a,b)$.

By definition $\psi[\phi(x)]$ is convex on (a,b).