Quiz 1, Statistical Inference I: Date 10/1 (2015 Fall)

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1. Let $X_1,..,X_n \overset{iid}{\sim} N(\xi,\sigma^2)$ and $X=(X_1,..,X_n)$.

1) Show that the joint density of $X=(X_1,..,X_n)$ forms an exponential family by specifying **all necessary components** (e.g., dimension s, natural parameters, parameter space, T functions, support, etc.).
2) Prove that the family is full rank.
3) Prove that the family is not full rank if $\xi=\sigma$ is assumed.

Ans:

1)

$X_1,..,X_n \overset{iid}{\sim} N(\xi,\sigma^2)$, $X=(X_1,..,X_n)$ let $\theta=(\xi,\sigma^2)$.

$$\begin{aligned}
P_\theta(x) &= \prod_{i=1}^{n} f_i(x_i) \\
&= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-1}{2\sigma^2}(x_i-\xi)^2 \right\} \\
&= \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i-\xi)^2 \right\} \\
&= \left(\frac{1}{2\pi} \right)^{\frac{n}{2}} \exp\left\{ \frac{\xi}{\sigma^2} \sum_{i=1}^{n} x_i - \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 - \frac{n}{2}\left(\frac{\xi^2}{\sigma^2} + \log\sigma^2 \right) \right\}.
\end{aligned}$$

Let

$$\eta_1(\theta)=\frac{\xi}{\sigma^2}, \quad \eta_2(\theta)=\frac{-1}{2\sigma^2}, \quad T_1(x)=\sum_{i=1}^{n} x_i, \quad T_2(x)=\sum_{i=1}^{n} x_i^2,$$

$$B(\theta)=\frac{n}{2}\left(\frac{\xi^2}{\sigma^2} + \log\sigma^2 \right) \text{ and } h(x)=\left(\frac{1}{2\pi} \right)^{\frac{n}{2}}.$$

Therefore,

$$P_\theta(x)=\exp\left\{ \sum_{i=1}^{n} \eta_i(\theta) T_i(x_i) - B(\theta) \right\} h(x)$$

is a two-dimensional exponential family with $\chi=(-\infty,\infty)^n$ and $\Theta=(-\infty,\infty)\times(-\infty,0)$.

2)

$$P_\theta(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left\{\frac{\xi}{\sigma^2}\sum_{i=1}^{n} x_i - \frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i{}^2 - \frac{n}{2}\left(\frac{\xi^2}{\sigma^2} + \log\sigma^2\right)\right\}.$$

$$\Theta = \left\{\left(\frac{\xi}{2\sigma^2}, \frac{-1}{2\sigma^2}\right); \quad \xi \in \mathrm{R}, \sigma^2 > 0\right\} = \{(\eta_1, \eta_2); \quad \eta_1 \in \mathrm{R}, \eta_2 < 0\}.$$

Θ contains a two-dimensional open rectangle hence (e.g., $\mathrm{A}(1,-1)$, $\mathrm{B}(1,-2)$, $\mathrm{C}(2,-1)$, $\mathrm{D}(2,-2)$ then ABCD is an two-dimensional open rectangle contained by Θ) hence it is full rank.

3)

Let $\xi = \sigma^2$.

$$P_\theta(x) = \left(\frac{1}{2\pi}\right)^{\frac{n}{2}} \exp\left\{\frac{1}{\xi}\sum_{i=1}^{n} x_i - \frac{1}{2\xi^2}\sum_{i=1}^{n} x_i{}^2 - \frac{n}{2}\left(1 + \log\xi^2\right)\right\}.$$

Since $\Theta = \left\{\left(\frac{1}{\xi}, \frac{-1}{2\xi^2}\right); \quad \xi > 0\right\} = \{(\eta_1, \eta_2); \quad \eta_1 > 0, \eta_2 < 0\}$.

Therefore, Θ does not contain a two-dimensional open rectangle hence it is not full rank.

2. Let $X_1, X_2 \overset{iid}{\sim} Poisson(\lambda)$ and $X = (X_1, X_2)$.

1) Derive the canonical form of the joint density of $X = (X_1, X_2)$ by specifying **all necessary components.**

<u>Ans:</u>

$$\begin{aligned} P_\lambda(x) &= \frac{1}{x_1!}\lambda^{x_1}e^{-\lambda} \cdot \frac{1}{x_2!}\lambda^{x_2}e^{-\lambda} \\ &= \frac{1}{x_1!x_2!}\lambda^{x_1+x_2}e^{-2\lambda} \\ &= \frac{1}{x_1!x_2!}\exp\{\log\lambda(x_1+x_2)-2\lambda\}. \end{aligned}$$

Let $\eta(\lambda) = \log\lambda$, we have

$$\begin{aligned} p(x\,|\,\eta) &= \frac{1}{x_1!x_2!}\exp\{\eta(\lambda)(x_1+x_2)-2e^{\eta}\} \\ &= \exp\{\eta(\lambda)T(x)-A(\eta)\}h(x) \end{aligned}$$

is the canonical form of a one-dimensional exponential family, where

$$\eta(\lambda) = \log\lambda,\ \ T(x) = x_1+x_2,\ \ A(\eta) = 2e^{\eta},\ \ h(x) = \frac{1}{x_1!x_2!},$$

$$\chi = \{0, 1, \cdots\}\times\{0, 1, \cdots\} \text{ and } \Theta = (0, \infty).$$

2) Prove that the family is full rank.

<u>Ans:</u>

By 1)

$$p(x\,|\,\eta) = \exp\{\eta(\lambda)T(x)-A(\eta)\}h(x).$$

Since $\Theta = \{\eta;\ \ \eta > 0\}$ contains an one-dimensional open rectangle (e.g., $(1, 2)$ is an one-dimensional open rectangle contained by Θ) hence it is full rank.

3. Let $p(x|\eta)=\exp\{\eta x - A(\eta)\}$, $x \in \mathrm{X}$, be a density w.r.t. the Lebesgue measure $\mu(x)=m(x)$. Derive the form of $A(\eta)$ and the natural parameter space Θ when $\mathrm{X}=[0,\infty)$.

<u>Ans:</u>

Since

$$p(x|\eta)=\exp\{\eta x - A(\eta)\}.$$

Hence

$$\int_\chi p(x|\eta)dx=1$$
$$\Rightarrow \int_\chi \exp\{\eta x - A(\eta)\}dx=1$$
$$\Rightarrow \int_\chi \exp\{\eta x\}dx=e^{A(\eta)}<\infty.$$

When $\mathrm{X}=[0,\infty)$, consider two different cases of $\eta>0$ and $\eta<0$.

$$\text{If } \eta>0 \Rightarrow \int_0^\infty \exp\{\eta x\}dx=\infty.$$

$$\text{If } \eta<0 \Rightarrow \int_0^\infty \exp\{\eta x\}dx<\infty$$

Therefore, the parameter space $\Theta=(-\infty,0)$.

Then

$$\int_0^\infty \exp\{\eta x\}dx=e^{A(\eta)}$$
$$\Rightarrow \left.\frac{1}{\eta}e^{\eta x}\right|_0^\infty=e^{A(\eta)}$$
$$\Rightarrow 0-\frac{1}{\eta}=e^{A(\eta)}$$
$$\Rightarrow -\frac{1}{\eta}=e^{A(\eta)}$$
$$\Rightarrow A(\eta)=\log\left(\frac{-1}{\eta}\right).$$

4. Let $X=(X_1,..,X_n)$, where $X_j \overset{iid}{\sim} f_\theta(x_j)=\exp\left\{\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x_j)-B'(\theta)\right\}h'(x_j)$, $j=1,..,n$. Show that the joint density of $X=(X_1,..,X_n)$ forms an exponential family by specifying **all necessary components** (e.g., dimension, natural parameters, parameter space, T functions, support, etc.).

<u>Ans:</u>

$X_j \overset{iid}{\sim} f_\theta(x_j)=\exp\left\{\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x_j)-B'(\theta)\right\}h'(x_j)$, $j=1,..,n$ and $X=(X_1,..,X_n)$.

$$
\begin{aligned}
P_\theta(x) &= \prod_{j=1}^{n} f_\theta(x_j)\\
&= \prod_{j=1}^{n}\exp\left\{\sum_{i=1}^{s}\eta_i'(\theta)T_i'(x_j)-B'(\theta)\right\}h'(x_j)\\
&= \exp\{\ \eta_1'(\theta)T_1'(x_1)+\eta_2'(\theta)T_2'(x_1)+\cdots+\eta_s'(\theta)T_s'(x_1)\\
&\qquad \eta_1'(\theta)T_1'(x_2)+\eta_2'(\theta)T_2'(x_2)+\cdots+\eta_s'(\theta)T_s'(x_2)\\
&\qquad\quad \vdots \qquad\qquad\qquad \vdots \qquad\qquad\qquad\qquad \vdots\\
&\qquad \eta_1'(\theta)T_1'(x_n)+\eta_2'(\theta)T_2'(x_n)+\cdots+\eta_s'(\theta)T_s'(x_n)-nB'(\theta)\ \}\prod_{j=1}^{n}h'(x_j)\\
&= \exp\left\{\eta_1'(\theta)\sum_{j=1}^{n}T_1'(x_j)+\eta_2'(\theta)\sum_{j=1}^{n}T_2'(x_j)+\cdots+\eta_s'(\theta)\sum_{j=1}^{n}T_s'(x_j)-nB'(\theta)\right\}\prod_{j=1}^{n}h'(x_j)\\
&= \exp\left\{\sum_{i=1}^{s}\left(\eta_i'(\theta)\sum_{j=1}^{n}T_i'(x_j)\right)-nB'(\theta)\right\}\prod_{j=1}^{n}h'(x_j)\\
&= \exp\left\{\sum_{i=1}^{s}\eta_i(\theta)T_i(x)-B(\theta)\right\}h(x)
\end{aligned}
$$

is a s-dimensional exponential family with

$$\eta_i(\theta)=\eta_i'(\theta),\ \ T_i(\theta)=\sum_{j=1}^{s}T_i'(x_j),\ \ B(\theta)=nB'(\theta),\ \ h(x)=\prod_{j=1}^{n}h'(x_j),$$

$$i=1,2,\cdots,s \quad \Theta=\left\{(\eta_1,\cdots,\eta_s);\ \int_\chi \exp\left\{\sum_{i=1}^{s}\eta_i(\theta)T_i(x)\right\}h(x)<\infty\right\} \text{ with}$$

support $\chi=\mathrm{R}^n$.