Homework#3 Statistical Inference I

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Problem 2.1

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with σ^2 is known. Find the UMVUE of **(a)** ξ^2, **(b)** ξ^3 and **(c)** ξ^4.

Solution:

(a)

Since $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ and σ^2 is known, we have

$$\begin{aligned} f(x_1, \cdots, x_n) &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{n} (x_i - \xi)^2 \right\} \\ &= \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 \right\} \exp\left\{ \frac{\xi}{\sigma^2} \sum_{i=1}^{n} x_i - \frac{n\xi^2}{2\sigma^2} \right\} \\ &= \exp\{ \eta T(x) - A(\eta) \} h(x), \end{aligned}$$

where

$$\eta = \frac{\xi}{\sigma^2}, \quad T(x) = \sum_{i=1}^{n} x_i, \quad A(\eta) = \frac{n\eta^2\sigma^2}{2}, \quad h(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ \frac{-1}{2\sigma^2} \sum_{i=1}^{n} x_i^2 \right\},$$

$$\chi \in R^n \quad \text{and} \quad \Theta = \{ \eta \mid \eta \in R \}.$$

Thus, it is the canonical form of an one-dimensional exponential family. Since the natural parameter space Θ contains an one-dimensional open rectangle (e.g., $(0,1) \in \Theta$) hence it is full rank. Therefore, $T(x) = \sum_{i=1}^{n} x_i$ is a complete sufficient statistic.

Let

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \sim N(\xi, \sigma^2/n).$$

Here I directly use the formula of $E(\overline{X}^k)$ in **Problem 2.4**. This formula will be proved latter. The formula is

$$E(\overline{X}^k) = \sum_{r=0}^{k}\binom{k}{r}\xi^{k-r}E(Y^r),$$

where

$$Y \sim N(0, \sigma^2/n)$$

and

$$E(Y^r) = \begin{cases} (r-1)(r-3)\cdots 3\cdot 1\cdot(\sigma^2/n)^{r/2} & \text{where } r \geq 2 \text{ is even} \\ 0 & \text{where } r \text{ is odd.} \end{cases}$$

Therefore,

$$\begin{aligned} E(\overline{X}^2) &= \sum_{r=0}^{2}\binom{2}{r}\xi^{2-r}E(Y^r) \\ &= \binom{2}{0}\xi^2 + \binom{2}{1}\xi E(Y) + \binom{2}{2}E(Y^2) \\ &= \xi^2 + \frac{\sigma^2}{n}. \end{aligned}$$

Hence we obtain

$$E\left(\overline{X}^2 - \frac{\sigma^2}{n}\right) = \xi^2.$$

Since $\overline{X}^2 - \frac{\sigma^2}{n}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^2 - \frac{\sigma^2}{n}$ is the UMVUE of ξ^2.

(b)

By the formula of $E(\overline{X}^k)$, we have

$$
\begin{aligned}
E(\overline{X}^3) &= \sum_{r=0}^{3} \binom{3}{r} \xi^{3-r} E(Y^r) \\
&= \binom{3}{0}\xi^3 + \binom{3}{1}\xi^2 E(Y) + \binom{3}{2}\xi E(Y^2) + \binom{3}{3}E(Y^3) \\
&= \xi^3 + \frac{3\xi\sigma^2}{n}.
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
E\left(\overline{X}^3 - \frac{3\sigma^2}{n}\overline{X}\right) &= E(\overline{X}^3) - \frac{3\sigma^2}{n}E(\overline{X}) \\
&= \xi^3 + \frac{3\xi\sigma^2}{n} - \frac{3\sigma^2}{n}\xi \\
&= \xi^3.
\end{aligned}
$$

Since $\overline{X}^3 - \frac{3\sigma^2}{n}\overline{X}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^3 - \frac{3\sigma^2}{n}\overline{X}$ is the UMVUE of ξ^3.

(c)

By the formula of $E(\overline{X}^k)$, we have

$$E(\overline{X}^4) = \sum_{r=0}^{4} \binom{4}{r} \xi^{4-r} E(Y^r)$$
$$= \binom{4}{0}\xi^4 + \binom{4}{1}\xi^3 E(Y) + \binom{4}{2}\xi^2 E(Y^2) + \binom{4}{3}\xi E(Y^3) + \binom{4}{4} E(Y^4)$$
$$= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2}.$$

Hence we obtain

$$E\left(\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}\right)$$
$$= E(\overline{X}^4) - \frac{6\sigma^2}{n} E(\overline{X}^2) + \frac{3\sigma^4}{n^2}$$
$$= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \frac{6\sigma^2}{n}\left(\xi^2 + \frac{\sigma^2}{n}\right) + \frac{3\sigma^4}{n^2}$$
$$= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \frac{6\sigma^2}{n}\xi^2 - \frac{6\sigma^4}{n^2} + \frac{3\sigma^4}{n^2}$$
$$= \xi^4.$$

Since $\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^4 - \frac{6\sigma^2}{n}\overline{X}^2 + \frac{3\sigma^4}{n^2}$ is the UMVUE of ξ^4.

Problem 2.2

(a)

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with both parameters are unknown. Then we can use the same method in **Problem 2.1** to show that

$$\left(\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i,\, S^2 = \sum_{i=1}^{n} (X_i - \overline{X}) \right)$$

is complete sufficient statistic.

Then we have

$$\frac{S^2}{\sigma^2} \sim \chi^2_{df=n-1}.$$

Therefore,

$$E\left(\frac{S^2}{\sigma^2}\right) = n-1 \Rightarrow E\left(\frac{S^2}{n(n-1)}\right) = \frac{\sigma^2}{n}.$$

By the previous results, we have

$$E(\overline{X}^2) = \xi^2 + \frac{\sigma^2}{n}.$$

Hence we obtain

$$\begin{aligned} E\left(\overline{X}^2 - \frac{S^2}{n(n-1)}\right) &= E(\overline{X}^2) - E\left(\frac{S^2}{n(n-1)}\right) \\ &= \xi^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \\ &= \xi^2. \end{aligned}$$

Since $\overline{X}^2 - \frac{S^2}{n(n-1)}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^2 - \frac{S^2}{n(n-1)}$ is the UMVUE of ξ^2.

(b)

Since $\overline{X}$ is complete sufficient statistic for ξ and S^2 is ancillary for ξ. By Basu's Theorem, $\overline{X}$ and S^2 are independent.

Therefore,

$$E\left(\frac{S^2}{\sigma^2}\overline{X}\right) = E\left(\frac{S^2}{\sigma^2}\right)E(\overline{X}) = (n-1)\xi$$
$$\Rightarrow E\left(\frac{3S^2}{n(n-1)}\overline{X}\right) = \frac{3\sigma^2\xi}{n}.$$

By the previous results, we have

$$E(\overline{X}^3) = \xi^3 + \frac{3\sigma^2\xi}{n}.$$

Hence we obtain

$$E\left(\overline{X}^3 - \frac{3S^2}{n(n-1)}\overline{X}\right) = E(\overline{X}^3) - E\left(\frac{3S^2}{n(n-1)}\overline{X}\right)$$
$$= \xi^3 + \frac{3\sigma^2\xi}{n} - \frac{3\sigma^2\xi}{n}$$
$$= \xi^3.$$

Since $\overline{X}^3 - \frac{3S^2}{n(n-1)}\overline{X}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^3 - \frac{3S^2}{n(n-1)}\overline{X}$ is the UMVUE of ξ^3.

(c)

Similarly,

$$E\left(\frac{S^2}{\sigma^2}\overline{X}^2\right)=E\left(\frac{S^2}{\sigma^2}\right)E(\overline{X}^2)=(n-1)\left(\xi^2+\frac{\sigma^2}{n}\right)$$
$$\Rightarrow E\left(\frac{6S^2}{n(n-1)}\overline{X}^2\right)=\frac{6\sigma^2\xi^2}{n}+\frac{6\sigma^4}{n^2},$$

$$E\left(\frac{S^4}{\sigma^4}\right)=\mathrm{var}\left(\frac{S^2}{\sigma^2}\right)+\left\{E\left(\frac{S^2}{\sigma^2}\right)\right\}^2=2(n-1)+(n-1)^2=(n-1)(n+1)=n^2-1$$
$$\Rightarrow E\left(\frac{3S^4}{n^2(n^2-1)}\right)=\frac{3\sigma^4}{n^2}.$$

By the previous results, we have

$$E(\overline{X}^4)=\xi^4+\frac{6\xi^2\sigma^2}{n}+\frac{3\sigma^4}{n^2}.$$

Hence we obtain

$$\begin{aligned}E\left(\overline{X}^4-\frac{6S^2}{n(n-1)}\overline{X}^2+\frac{3S^4}{n^2(n^2-1)}\right)&=E(\overline{X}^4)-E\left(\frac{6S^2}{n(n-1)}\overline{X}^2\right)+E\left(\frac{3S^4}{n^2(n^2-1)}\right)\\&=\xi^4+\frac{6\xi^2\sigma^2}{n}+\frac{3\sigma^4}{n^2}-\frac{6\sigma^2\xi^2}{n}-\frac{6\sigma^4}{n^2}+\frac{3\sigma^4}{n^2}\\&=\xi^4.\end{aligned}$$

Since $\overline{X}^4-\frac{6S^2}{n(n-1)}\overline{X}^2+\frac{3S^4}{n^2(n^2-1)}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^4-\frac{6S^2}{n(n-1)}\overline{X}^2+\frac{3S^4}{n^2(n^2-1)}$ is the UMVUE of ξ^3.

Problem 2.4

(a)

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with σ^2 is known. The formula of $E(\overline{X}^k)$ is

$$E(\overline{X}^k) = \sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} E(Y^r),$$

where

$$Y \sim N(0, \sigma^2 / n)$$

and

$$E(Y^r) = \begin{cases} (r-1)(r-3)\cdots 3\cdot 1\cdot(\sigma^2/n)^{r/2} & \text{where } r \geq 2 \text{ is even} \\ 0 & \text{where } r \text{ is odd.} \end{cases}$$

Proof:

One can write $\overline{X} = Y + \xi$. By the binomial theorem, we have

$$\begin{aligned} E(\overline{X}^k) &= E\{(Y+\xi)^k\} \\ &= E\left\{\sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} Y^r\right\} \\ &= \sum_{r=0}^{k} \binom{k}{r} \xi^{k-r} E(Y^r). \end{aligned}$$

Then

$$E(Y^r) = \int_{-\infty}^{\infty} y^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{\frac{-y^2}{2\sigma^2/n}\right\} dy.$$

Using the change of variable $y = \frac{\sigma}{\sqrt{n}} u$, then we can obtain

$$\int_{-\infty}^{\infty} y^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{\frac{-y^2}{2\sigma^2/n}\right\} dy$$
$$= \int_{-\infty}^{\infty} \left(\frac{\sigma}{\sqrt{n}} u\right)^r \frac{1}{\sqrt{2\pi\sigma^2/n}} \exp\left\{\frac{-1}{2\sigma^2/n}\frac{\sigma^2}{n}u^2\right\} \frac{\sigma}{\sqrt{n}} du$$
$$= \left(\frac{\sigma}{\sqrt{n}}\right)^r \int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du.$$

Then consider the integral

$$\int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du .$$

If r is odd, it is an integral of an odd function over a real line. Hence it is zero. If r is even, consider the change of variable $u = \sqrt{w}$, then we have

$$\begin{aligned}
\int_{-\infty}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du &= 2\int_{0}^{\infty} u^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-u^2}{2}\right\} du \\
&= 2\int_{0}^{\infty} (\sqrt{2w})^r \frac{1}{\sqrt{2\pi}} \exp\left\{\frac{-(\sqrt{2w})^2}{2}\right\} \frac{1}{\sqrt{2w}} dw \\
&= \frac{2^{\frac{r}{2}}}{\sqrt{\pi}} \int_{-\infty}^{\infty} w^{\frac{r}{2}-\frac{1}{2}} e^{-w} dw = \frac{2^{\frac{r}{2}}}{\sqrt{\pi}} \Gamma\left(\frac{r}{2}+\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{\pi}} 2^{\frac{r}{2}} \left(\frac{r-1}{2}\right)\left(\frac{r-3}{2}\right) \cdots 3 \cdot 1 \cdot \sqrt{\pi} \\
&= (r-1)(r-3)\cdots 3 \cdot 1.
\end{aligned}$$

Therefore, we obtain

$$E(Y^r) = \begin{cases} (r-1)(r-3)\cdots 3\cdot 1\cdot(\sigma^2/n)^{r/2} & \text{where } r \ge 2 \text{ is even} \\ 0 & \text{where } r \text{ is odd.} \end{cases}$$

Hence we have shown the formula of $E(\overline{X}^k)$.

(b)

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with ξ is known. Define

$$S^2 = \sum_{i=1}^{n} (X_i - \xi)^2 .$$

We can use the same method in **Problem 2.1** to show that S^2 is an complete sufficient statistic. Moreover, we have

$$\frac{S^2}{\sigma^2} \sim \chi^2_{df=n}.$$

Therefore,

$$E\left(\frac{S^4}{\sigma^4}\right) = \text{var}\left(\frac{S^2}{\sigma^2}\right) + \left\{E\left(\frac{S^2}{\sigma^2}\right)\right\}^2 = 2n + n^2 = n(n+2)$$
$$\Rightarrow E(S^4) = n(n+2)\sigma^4,$$

$$\text{var}\left(\frac{S^2}{\sigma^2}\right) = 2n \Rightarrow \text{var}\left(\frac{S^2}{n}\right) = \frac{2\sigma^4}{n}.$$

Since

$$E(S^4) = n(n+2)\sigma^4.$$

Then we can obtain

$$E\left(\frac{2S^4}{n^2(n+2)}\right) = \frac{2\sigma^4}{n}.$$

Since $\frac{2S^4}{n^2(n+2)}$ is a function of complete sufficient statistic. Therefore, $\frac{2S^4}{n^2(n+2)}$ is the UMVUE of $\frac{2\sigma^4}{n}$.

Problem 2.5

If $X_1, \cdots, X_n \overset{iid}{\sim} N(\xi, \sigma^2)$ with both parameters are unknown. Then we can use the same method in **Problem 2.1** to show that

$$\left(\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, S^2 = \sum_{i=1}^{n} (X_i - \overline{X}) \right)$$

is complete sufficient statistic.

Now we have

$$\frac{S^2}{\sigma^2} \sim \chi^2_{df=n-1}.$$

Therefore,

$$E\left(\frac{S^2}{\sigma^2}\right) = n-1 \Rightarrow E\left(\frac{S^2}{n(n-1)}\right) = \frac{\sigma^2}{n}.$$

By the previous results, we have

$$E(\overline{X}^2) = \xi^2 + \frac{\sigma^2}{n}.$$

Hence we obtain

$$\begin{aligned} E\left(\overline{X}^2 - \frac{S^2}{n(n-1)}\right) &= E(\overline{X}^2) - E\left(\frac{S^2}{n(n-1)}\right) \\ &= \xi^2 + \frac{\sigma^2}{n} - \frac{\sigma^2}{n} \\ &= \xi^2. \end{aligned}$$

Since $\overline{X}^2 - \frac{S^2}{n(n-1)}$ is a function of complete sufficient statistic. Therefore, $\overline{X}^2 - \frac{S^2}{n(n-1)}$ is the UMVUE of ξ^2.

Problem 2.6

(a)

Here $\overline{X}$ and S^2 are the same notations as in **Problem 2.5**. Since $\overline{X}$ is complete sufficient statistic for ξ and S^2 is ancillary for ξ. By Basu's Theorem, $\overline{X}$ and S^2 are independent.

Since

$$\begin{aligned}
\mathrm{var}(\overline{X}^2) &= E(\overline{X}^4) - \{E(\overline{X}^2)\}^2 \\
&= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \left\{\xi^2 + \frac{\sigma^2}{n}\right\}^2 \\
&= \xi^4 + \frac{6\xi^2\sigma^2}{n} + \frac{3\sigma^4}{n^2} - \xi^4 - \frac{\sigma^4}{n^2} - \frac{2\xi^2\sigma^2}{n} \\
&= \frac{4\xi^2\sigma^2}{n} + \frac{2\sigma^4}{n^2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
\mathrm{var}\left(\overline{X}^2 - \frac{S^2}{n(n-1)}\right) &= \mathrm{var}(\overline{X}^2) + \mathrm{var}\left(\frac{S^2}{n(n-1)}\right) \\
&= \frac{4\xi^2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \frac{\sigma^4}{n^2(n-1)^2}\mathrm{var}\left(\frac{S^2}{\sigma^2}\right) \\
&= \frac{4\xi^2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \frac{2\sigma^4}{n^2(n-1)}.
\end{aligned}$$

Hence we obtain

$$\mathrm{var}\left(\overline{X}^2 - \frac{S^2}{n(n-1)}\right) = \frac{4\xi^2\sigma^2}{n} + \frac{2\sigma^4}{n^2} + \frac{2\sigma^4}{n^2(n-1)}.$$