Statistical Inference I

Midterm exam: 2012/11/13(Tue)
9:00-12:00, Room M708

Not open book.

Q1

Q2

Q3

Q4

Q5

YOUR NAME______________________

NOTE1: Please write down the derivation of your answer very clearly for all questions. The score will be reduced when you only write answer. Also, the score will be reduced if the derivation is not clear. The score will be added even when your answer is incorrect but the derivation is correct.

1. Let $(X_1,\ldots,X_n)$ be iid from a distribution with $E[X_i]=\mu$ and $Var[X_i]=\sigma^2$. Also, let $\bar{X}$ be the sample mean and $S^2=\sum_{i=1}^{n}(X_i-\bar{X})^2/(n-1)$ be the sample variance. Consider estimation of μ^2 under a square error loss.

   a) Calculate $E(S^2)$.

   b) Calculate the bias of $T_1=\bar{X}^2$.

   c) Calculate the bias of $T_2=\bar{X}^2-S^2/n$.

   d) Give a distribution of X_i such that T_2 is the UMVUE

   e) Give a distribution of X_i such that T_2 is **not** the UMVUE

   f) In the above case, what is the UMVUE?

2. Let $\Im = \{ T_j : j = 1, ..., 8 \}$ be a class of decision rules. It is known that a) T_4 is better than T_1; b) T_1 is better than T_5; c) T_6 is better than T_7; d) T_2 is better than T_3. Otherwise, there is no other relationship between T_j and T_k ($j \neq k$). Find all admissible rules.

3. Let $(X_1, ..., X_n)$ be iid samples from $N(\mu, \sigma^2)$, where (μ, σ^2) are unknown. Consider estimation of σ^2 under the squared error loss. Is $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})$ admissible? If so, give a proof. If not, give a counterexample.

4. **Lehmann-Scheffe's theorem**

Let $T(X)$ be a complete and sufficient statistic for $P \in \mathrm{P}$. Also, let ϑ be a parameter related to P, which is estimable (at least one unbiased estimator exists). Then, by Lehmann-Sheffe's theorem, there exist a unique UMVUE of the form $h\{T(X)\}$ where h is a Borel function. Please give a proof.

5. Random vectors $\mathbf{X}_i = \begin{bmatrix} X_{i1} \\ X_{i2} \end{bmatrix}$, $i = 1,\ldots,n$ follow (iid) bivariate normal distribution

$$f_{\boldsymbol{\mu},\boldsymbol{\Sigma}}(\mathbf{x}) = \frac{1}{2\pi |\boldsymbol{\Sigma}|^{1/2}} \exp\left\{-\frac{1}{2}(\mathbf{x}-\boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu})\right\},$$

where $\boldsymbol{\mu} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}$ and $\boldsymbol{\Sigma} = \begin{bmatrix} \sigma_{11}^2 & \sigma_{12} \\ \sigma_{12} & \sigma_{22}^2 \end{bmatrix}$, and the parameter $\boldsymbol{\theta} = (\mu_1, \mu_2)^T$ is unknown.

1) Find the UMVUE of $\vartheta = \mu_1 - \mu_2$ and its variance.
2) Find the Fisher information matrix.
3) Find the Cramer-Rao lower bound for $\vartheta = \mu_1 - \mu_2$.

Answer 1

a) $E[S^2] = \frac{n}{n-1}\left(E[X_i^2] - E[\overline{X}^2]\right) = \frac{n}{n-1}\left(\mu^2 + \sigma^2 - (\frac{\sigma^2}{n} + \mu^2)\right)$

$= \frac{\sigma^2}{n}$

b) $E[T_1] = \frac{1}{n^2} E\left\{\sum_i X_i^2 + 2\sum_{i<j} X_i\, X_j\right\} = \frac{n(\mu^2 + \sigma^2) + n(n-1)\mu^2}{n^2} = \mu^2 + \frac{\sigma^2}{n}$.

Or $E[T_1] = E[\overline{X}^2] = Var[\overline{X}] + (E[\overline{X}])^2 = \mu^2 + \frac{\sigma^2}{n}$.

$b_{T_1}(P) = \frac{\sigma^2}{n}$.

c) Unbiased.

d) For example, if $X = N(\mu, \sigma^2)$, then $T_2 = \overline{X}^2 - S^2/n$ is a function of the CSS $(\overline{X}, S^2)$. By Lehmann-Scheffe theorem, T_2 is unique UMVUE of μ^2.

e) If $X = Poisson(\mu)$, $\overline{X}$ is CSS. Notice that $E[\overline{X}^2] = \mu^2 + \frac{\mu^2}{n} = \frac{n+1}{n}\mu^2$.

Hence, $T_3 = n/(n+1)\overline{X}^2$ is the UMVUE.

Answer 2 T_2, T_4, T_6, T_8.

Answer 3

Inadmissible: $(n-1)S^2/n$ is better than $S^2 = 1/(n-1)\sum_{i=1}^{n}(X_i - \overline{X})$.

$E[S^2 - \sigma^2]^2 = 2\sigma^4/(n-1)$

$E[(n-1)S^2/n - \sigma^2]^2 = \sigma^4(2n-1)/n^2 < 2\sigma^4/(n-1)$.

Answer 5

1) $T = \overline{X} - \overline{Y}$, $Var(T) = \frac{\sigma_{11}^2 + \sigma_{22}^2 - 2\sigma_{12}}{n}$

2) $I_1(\boldsymbol{\theta}) = \begin{bmatrix} \sigma_{22}^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_{11}^2 \end{bmatrix} = \boldsymbol{\Sigma}^{-1}$, $I(\boldsymbol{\theta}) = n\boldsymbol{\Sigma}^{-1}$

3) $(1,-1)\frac{\boldsymbol{\Sigma}}{n}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = \frac{\sigma_{11}^2 + \sigma_{22}^2 - 2\sigma_{12}}{n}$.