Statistical inference I, 2012 Fall, Homework#3

1. $X_1,..,X_n \overset{iid}{\sim} N(\mu,\sigma^2)$, $\theta=(\mu,\sigma^2)$. Let $\vartheta=\mu^2$ and $T_n=\bar{X}^2$ be an estimator of ϑ. Find $mse_{T_n}(\theta)=E(T_n-\vartheta)^2$ and compare it with the asymptotic MSE.

2. $X=(X_1,..,X_n)\overset{iid}{\sim} N(\mu,1)$, where $\mu\in[c,d]=\Theta$ and let

$T_1(X)=cI(\bar{X}<c)+\bar{X}I(c\le\bar{X}\le d)+dI(\bar{X}>d)$ be an estimator of μ.

a) Calculate $F_{T_1}(x)=P(T_1(X)\le x)$ using the standard normal c.d.f. $\Phi(x)$.

b) If c >0, show that $E[T_1(X)]=\int_0^\infty[1-F_{T_1}(x)]dx$.

c) Obtain $E[T_1(X)]$ in terms of an integration of $\Phi(x)$.

3. $X=(X_1,..,X_n)\overset{iid}{\sim} N(\mu,1)$, where $\mu\in R$. Let $\vartheta=P(X_1\le c)=\Phi(c-\mu)$, where Φ is the c.d.f. of $N(0,1)$ and c is a fixed constant. Let $T_{1n}=F_n(c)$, where F_n is the empirical c.d.f., and $T_{2n}=\Phi(c-\bar{X})$.

a) Find the asymptotic MSE's of T_{1n} ad T_{2n}.

b) Which estimator is asymptotically more efficient?

3. Exercise 122(2nd ed., J. Shao, p.159):

$X_1,..,X_1 \overset{iid}{\sim} Bin(n=1,p)$. Let a and b be positive constants. Find the asymptotic relative efficiency of $(a+n\bar{X})/(a+b+n)$ w.r.t. $\bar{X}$.

This is a simplified answer. In the exam, you need to write more detailed calculations.

Answer 1

Note that $\frac{n(\overline{X}-\mu)^2}{\sigma^2} \sim \chi^2_{df=1}$. This leads to

$E\{(\overline{X}-\mu)^2\} = \sigma^2/n$ and $Var\{(\overline{X}-\mu)^2\} = 2\sigma^4/n^2$. Hence,

$E(\overline{X}-\mu)^4 = Var\{(\overline{X}-\mu)^2\} + \{E(\overline{X}-\mu)^2\}^2 = 3\sigma^4/n^2$.

Therefore,

$$
\begin{aligned}
& mse_{T_n}(\theta) \\
&= E(\overline{X}^2-\mu^2)^2 = E\{(\overline{X}-\mu+\mu)^2-\mu^2\}^2 = E\{(\overline{X}-\mu)^2+2\mu(\overline{X}-\mu)\}^2 \\
&= E(\overline{X}-\mu)^4 + 4\mu E(\overline{X}-\mu)^3 + 4\mu^2 E(\overline{X}-\mu)^2 \\
&= 3\sigma^4/n^2 + 4\mu^2\sigma^2/n
\end{aligned}
$$

i)If $\mu \neq 0$, By delta method, $\sqrt{n}(\overline{X}^2-\mu^2) \to_d N(0, 4\mu^2\sigma^2)$.

$amse_{T_n}(\theta) = 4\mu^2\sigma^2/n$.

ii) If $\mu = 0$, $\sqrt{n}\overline{X}/\sigma = N(0,1)$, and $n\overline{X}^2/\sigma^2 = \chi^2_{df=1}$. Hence,

$n^2 E[\overline{X}^4]/\sigma^4 = E[(\chi^2_{df=1})^2] = 3$, and $amse_{T_n}(\theta) = E[\overline{X}^4] = 3\sigma^4/n^2$.

Answer.

- $mse_{T_n}(\theta) = 3\sigma^4/n^2 + 4\mu^2\sigma^2/n$
- $amse_{T_n}(\theta) = \begin{cases} 4\mu^2\sigma^2/n & (\mu \neq 0) \\ 3\sigma^4/n^2 & (\mu = 0) \end{cases}$

Answer 2

a) $F_{T_1}(x) = \begin{cases} 1 & (x > d) \\ \Phi\{\sqrt{n}(x-\mu)\} & (c \le x \le d) \\ 0 & (x < c) \end{cases}$.

b)

$$
\begin{aligned}
&\int_0^\infty [1-F_{T_1}(x)]dx = \int_0^\infty P(T_1(X) > x)dx = \int_0^\infty \int_x^\infty dF_{T_1}(y)dx \\
&= \int_0^\infty \int_0^y dx dF_{T_1}(y) = \int_0^\infty y dF_{T_1}(y) = E[T_1(X)].
\end{aligned}
$$

c) $E[T_1(X)] = \int_0^\infty [1-F_{T_1}(x)]dx = d - \int_c^d \Phi\{\sqrt{n}(x-\mu)\}dx = d - \frac{1}{\sqrt{n}} \int_{\sqrt{n}(c-\mu)}^{\sqrt{n}(d-\mu)} \Phi(t)dt$.

Answer 3

Since $\sqrt{n}(T_{1n} - \vartheta) \rightarrow_d N(0, \vartheta(1-\vartheta))$,

$amse_{T_{1n}}(\mu) = \vartheta(1-\vartheta)/n$.

By $\sqrt{n}(\bar{X} - \mu) \rightarrow_d N(0,1)$ and the delta method for $g(x) = \Phi(c-x)$,

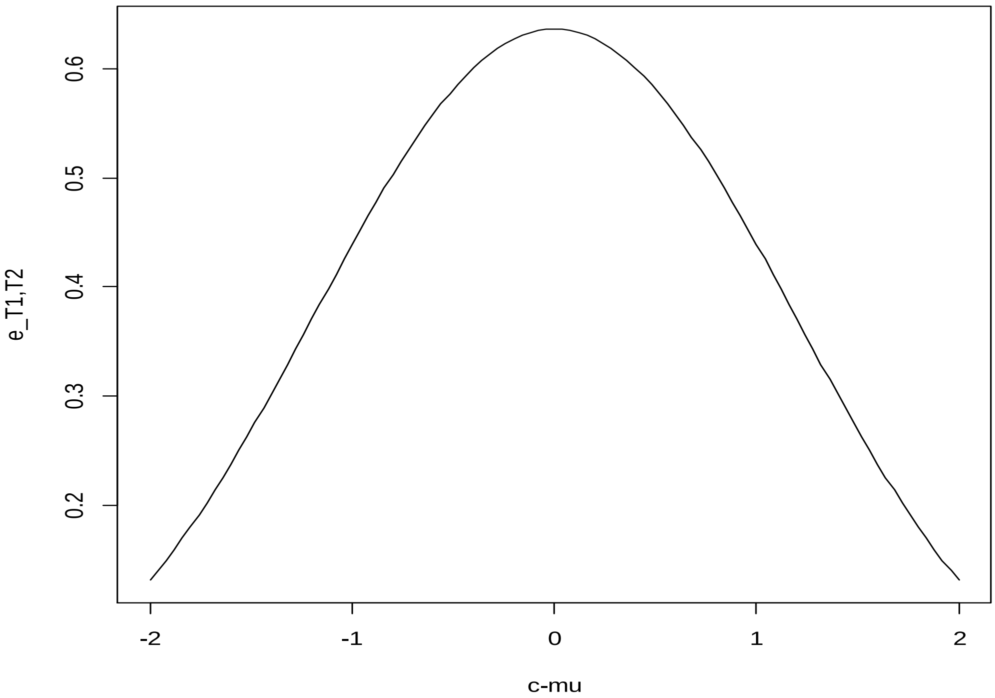
$\sqrt{n}(T_{2n} - \vartheta) \rightarrow_d N(0, \phi^2(c-\mu))$.

Hence, the asymptotic relative efficiency of T_{1n} w.r.t. T_{2n} is

$amse_{T_{2n}}(\mu) = \phi^2(c-\mu)/n$.

$$e_{T_{1n},T_{2n}}\mu(\mu) = \frac{\phi^2(c-\mu)}{\Phi(c-\mu)\{1-\Phi(c-x)\}} < 1 \text{. (see Figure)}$$

Hence, T_{2n} is asymptotically more efficient than T_{1n}.




```
func=function(u){
dnorm(u)^2/pnorm(u)/(1-pnorm(u))
}
curve(func,-2,2,xlab="c-mu",ylab="e_T1,T2")
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Answer 4

By CLT, $\sqrt{n}(\overline{X} - p) \to_d Y = N(0, p(1-p))$.

Let $T(X) = (a + n\overline{X})/(a+b+n)$. Then,

$$
\begin{aligned}
&\sqrt{n}(T(X) - p) \\
&= a\sqrt{n}/(a+b+n) + \sqrt{n}\{n\overline{X}/(a+b+n) - p\} \\
&= o(1) - \overline{X}\left\{\sqrt{n}(a+b)/(a+b+n)\right\} + \sqrt{n}(\overline{X} - p) \\
&= o_P(1) + \sqrt{n}(\overline{X} - p) \\
&\to_d Y
\end{aligned}
$$

$amse_T(p) = amse_{\overline{X}}(p) = p(1-p)/n$.

Hence, ARE is 1.□