Statistical inference I, 2012 Fall, Homework #2

1. Sigma-field:

Let $X:(\Omega,\mathbf{F},P)\to(R,\mathbf{B})$ be a random variable, representing the failure time of an electric component.

(i) Suppose an engineer *cannot* observe the exact failure time X. Instead, he/she can observe the failure status of the electric component at time t>0. That is, the engineer can identify whether the event $\{X\le t\}$ occurs or not. Show that $\{X\le t\}\in\mathbf{F}$ (just recall the definition of random variables in my note or p.7 of the text).

(ii) Find $\sigma(\{X\le t\})$, a σ-field generated by the event $\{X\le t\}$.

(this σ-field represents all events that the engineer can observe at time t)

(iii) Suppose that an engineer can observe the failure status of the electric component at two different times $t_1<t_2$. Find $\sigma(\{X\le t_1\},\{X\le t_2\})$.

(iv) Suppose that an engineer can observe the failure status of the electric component at $t_1<t_2<\cdots<t_k$. Explicitly write

$\mathbf{F}_k=\sigma([\{X\le t_j\};\, j=1,2,...,k])$. How many elements are there?

($\mathbf{F}_j$ represents all events that the engineer can observe before t_k)

(v) Show that $\mathbf{F}_{i-1}\subset\mathbf{F}_i$ for $i=2,\ldots,k$.

(vi) Prove or disprove $\{X>t_i\}\in\mathbf{F}_i$, $\{X>t_i\}\in\mathbf{F}_{i-1}$, and $\{X\ge t_i\}\in\mathbf{F}_{i-1}$

2. Asymptotic analysis

Show that $X_n=o_P(Y_n)$ implies $X_n=O_P(Y_n)$.

3. Exponential family

Problem 13 (p.143), Problem 14 (144)

4. Location-scale family

Suppose that a probability measure $P_{\boldsymbol{\mu},\boldsymbol{\Sigma},\nu}$ defined on the k-dimensional Borel field has a density (w.r.t. a Lebesgue measure)

$$f_{\boldsymbol{\mu},\boldsymbol{\Sigma},\nu}(\mathbf{x})=\frac{\det(\Sigma)^{-1/2}\Gamma\{(\nu+k)/2\}}{\Gamma(1/2)^k\ \Gamma(\nu/2)\nu^{k/2}}\left\{1+\frac{(\mathbf{x}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{x}-\boldsymbol{\mu}}{\nu}\right\}^{-(\nu+k)/2}.$$

(i) Show that $P_{\boldsymbol{\mu},\boldsymbol{\Sigma},\nu}(B)=P_\nu(\boldsymbol{\Sigma}^{-1/2}(B-\boldsymbol{\mu}\boldsymbol{\Sigma}))$ for some P_ν, and write down P_ν.

(ii) For cases of $k=1,2$, show $P_\nu\to_d P$ using Scheffe's theorem (p.59). Please explain in details how to apply the Scheffe's theorem.

NOTE: This is my simplified answers. You need to write more detailed calculations.

Answer 1

(ii) $\sigma(\{X \le t\}) = [\phi, \{X \le t\}, \{X > t\}, \Omega]$.

(iii) $$\sigma(\{X \le t_1\}, \{X \le t_2\})$$
$$= [\phi, \{X \le t_1\}, \{X \le t_2\}, \{X > t_1\}, \{X > t_2\}, \{t_1 < X \le t_2\}, \{X \le t_1 \text{ or } X > t_2\}, \Omega]\,.$$

8 elements.

(iv) Let $t_0 = -\infty$. Then,

$$\mathbf{F}_k = \sigma([\{X \le t_j\};\ j = 1,2,\dots,k])$$
$$= \left\{ \bigcup_{j=1}^{k+1} A_j \,\middle|\, A_j = \{X \in (t_{j-1}, t_j]\} \text{ or } \phi,\ j = 1,\dots,k; \quad A_{k+1} = \{X > t_k\} \text{ or } \phi \right\}$$

Hence, the number of elements is 2^{k+1}. I check this number by a special case of k=3 as follows:

$\sigma(\{X \le t_1\}, \{X \le t_2\}, \{X \le t_3\}) =$
$[\phi, \Omega,$
$\{X \le t_1\}, \{X \le t_2\}, \{X \le t_3\},$
$\{X > t_1\}, \{X > t_2\}, \{X > t_3\}$
$\{t_1 < X \le t_2\}, \{t_2 < X \le t_3\}, \{t_2 < X \le t_3\}, \{t_{`} < X \le t_3\},$
$\{t_1 < X \le t_2\}^c, \{t_2 < X \le t_3\}^c, \{t_2 < X \le t_3\}^c, \{t_{`} < X \le t_3\}^c]$

Hence, we have 16= 2^{3+1} elements.

(v) Omit [it follows from (A1)].

(vi) From (A1), $\{X > t_i\} \in \mathbf{F}_i$, $\{X > t_i\} \notin \mathbf{F}_{i-1}$ and $\{X \ge t_i\} = \{X > t_{i-1}\} \in \mathbf{F}_{i-1}$.