Statistical Inference I

Midterm exam: 2011/11/28(Mon)

Q1
Q2
Q3
Q4
Q5
Q6

YOUR NAME______________________

NOTE1: Please write down the derivation of your answer very clearly for all questions. The score will be reduced when you only write answer. Also, the score will be reduced if the derivation is not clear. The score will be added even when your answer is incorrect but the derivation is correct.

1. Ref: Exercise 61

Let $(X_1,..., X_n)$ be iid from U(*a*, *b*), where $a < b$.

(a) Show that ($X_{(1)} = \min_i(X_i)$, $X_{(n)} = \max_i(X_i)$) is sufficient for (*a*, *b*).

(b) It is known (by Exercise 51) that ($X_{(1)} = \min_i(X_i)$, $X_{(n)} = \max_i(X_i)$) is complete for (*a*, *b*). Please show that $(X_i - X_{(1)})/(X_{(n)} - X_{(1)})$, *i=2,..,n-1* are independent of $(X_{(1)}, X_{(n)})$.

2. Ref: Exercise 63

Let $(X_1,..., X_n)$ be iid from $N(\mu, \sigma^2)$. Consider the estimation of σ^2 with the squared error loss. Show that $(n-1)S^2/n$ is better than $S^2 = 1/(n-1)\sum_{i=1}^{n}(X_i - \overline{X})$.

3. **Lehmann-Scheffe's theorem**

Let $T(X)$ be a complete and sufficient statistic for $P \in \mathrm{P}$. Also, let ϑ be a parameter related to P, which is estimable (at least one unbiased estimator exists). Then, by Lehmann-Sheffe's theorem, there exist a unique UMVUE of the form $h\{T(X)\}$ where h is a Borel function. Please give a proof.

4. Ref: Exercise 64

Let $X=(X_1,...,X_n)$ be iid binary random variables with $\Pr(X_i=1)=\theta$. Consider estimating θ with the squared error loss. Calculate the risks of the following estimators:

(a) Sample mean $\bar{X}$

(b) $T_0(X)=\begin{cases} 0 & \text{if more than half of } X_i\text{'s are } 0 \\ 1 & \text{if more than half of } X_i\text{'s are } 1 \\ 0.5 & \text{if exactly half of } X_i\text{'s are } 0 \end{cases}$

NOTE: Consider two cases: even (*n=2k* for a natural number *k*) and odd (*n=2k+1* for a natural number *k*), and write your answer in terms of θ and k.

(c) The randomized estimator $T_1(X)=\begin{cases} \bar{X} & \text{with probability } 1/2 \\ T_0 & \text{with probability } 1/2 \end{cases}$, where T_0 is a constant.

(d) The randomized estimator $T_2(X)=\begin{cases} \bar{X} & \text{with probability } \bar{X} \\ 1/2 & \text{with probability } 1\text{-}\bar{X} \end{cases}$

5. Ref. Exercise 67

Let $X = (X_1,...,X_n)$ be iid having the exponential distribution $\Pr(X_i \le t) = 1 - \exp(-t/\theta)$. Consider the hypothesis

$$H_0 : \theta \le \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0 .$$

Obtain the risk function of the test rule $T_c(X) = I_{(c,\infty)}(\bar{X})$ under the 0-1 loss.

6. Two sample problem

Consider a linear model

$$X_{ij} = \mu + \alpha_i + \varepsilon_{ij}, \quad i = 1,2;\ j = 1,...,n_i .$$

where $\beta = (\mu, \alpha_1, \alpha_2)^T$ is a vector of unknown parameters.

a) Write down the above model in the form of $X = Z\beta + \varepsilon$.

b) Is $Z^T Z$ of full rank?

c) Find a generalized inverse $(Z^T Z)^-$.

d) Show that $(Z^T Z)^-$ is not unique.

Answer 1

a) By factorization theorem.

b) Let $Z_i = (X_i - a)/(b-a) \sim U(0,1)$. Then, apply the Basu's theorem.

Answer 2

$E[S^2 - \sigma^2]^2 = 2\sigma^4/(n-1)$

$E[(n-1)S^2/n - \sigma^2]^2 = \sigma^4(2n-1)/n^2 < 2\sigma^4/(n-1)$.

Answer 6

a) $X = Z\beta + \varepsilon$, where $X = (X_{11},..., X_{1n_1}, X_{21},..., X_{2n_2})^T$

$$\varepsilon = (\varepsilon_{11},..., \varepsilon_{1n_1}, \varepsilon_{21},..., \varepsilon_{2n_2})^T, \quad Z = \begin{bmatrix} 1 & 1\,0 \\ \vdots & \vdots\,\vdots \\ 1 & 1\,0 \\ 1 & 0\,1 \\ \vdots & \vdots\,\vdots \\ 1 & 0\,1 \end{bmatrix},$$

b) $Z^T Z = \begin{bmatrix} n & n_1 & n_2 \\ n_1 & n_1 & 0 \\ n_2 & 0 & n_2 \end{bmatrix}$, rank(Z)=2, not full rank

c) An LSE satisfies $Z^T Z\beta = Z^T X$. Consider an estimator of $\beta = (\mu, \alpha_1, \alpha_2)^T$ as

$\hat{\beta} = \begin{bmatrix} \bar{X} \\ \bar{X}_1 - \bar{X} \\ \bar{X}_3 - \bar{X} \end{bmatrix} = \begin{bmatrix} 1/n & 0 & 0 \\ -1/n & 1/n_1 & 0 \\ -1/n & 0 & 1/n_2 \end{bmatrix} Z^T X$. Hence, we may set

$(Z^T Z)^- = \begin{bmatrix} 1/n & 0 & 0 \\ -1/n & 1/n_1 & 0 \\ -1/n & 0 & 1/n_2 \end{bmatrix}$. Indeed, it holds that $(Z^T Z)(Z^T Z)^-(Z^T Z) = (Z^T Z)$.

d) Take $(Z^T Z)^- = \begin{bmatrix} a/n & 0 & 0 \\ -a/n & 1/n_1 & 0 \\ -a/n & 0 & 1/n_2 \end{bmatrix}$, $a \in \mathbf{R}$.