Homework#4, Statistical Inference I, 2011 Fall

1. Residual in LSE

Consider a linear model $X = Z\beta + \varepsilon$, where $\varepsilon \sim N_n(0, \sigma^2 I_n)$. Let $\hat{\beta} = (Z^T Z)^- Z^T X$ be an LSE. Please give a proof for

$$E\left\| X - Z\hat{\beta} \right\| = \sigma^2 [n - \mathrm{tr}\{(Z^T Z)^- Z^T Z\}].$$

2. UMP test

Let $Y \sim Bin(n=5, p)$.

1) Please derive a UMP test for $H_0 : p = 1/3$ vs. $H_1 : p = 2/3$ of level $\alpha = 0.05$.
2) Study Lemma 6.1 (p.397), and then derive a UMP test for $H_0 : p = 1/3$ vs. $H_1 : p > 1/3$.

<u>1.Answer</u>

Note that

$$\mathrm{tr}\{Var(Z\hat{\beta})\} = \mathrm{tr}\{Z Var(\hat{\beta}) Z^T\} = \sigma^2 \mathrm{tr}\{Z(Z^T Z)^- Z^T Z (Z^T Z)^- Z^T\}$$
$$= \sigma^2 \mathrm{tr}\{(Z^T Z)^- Z^T Z (Z^T Z)^- Z^T Z\} = \sigma^2 \mathrm{tr}\{(Z^T Z)^- Z^T Z\}$$

Hence,

$$E\left\| X - Z\hat{\beta} \right\|^2 = E\left\| X - Z\beta \right\|^2 - E\left\| Z\hat{\beta} - Z\beta \right\|^2$$
$$= E[\mathrm{tr}\{(X - Z\beta)^T (X - Z\beta)\}] - E[\mathrm{tr}\{(Z\hat{\beta} - Z\beta)^T (Z\hat{\beta} - Z\beta)\}]$$
$$= E[\mathrm{tr}\{(X - Z\beta)(X - Z\beta)^T\}] - E[\mathrm{tr}\{(Z\hat{\beta} - Z\beta)(Z\hat{\beta} - Z\beta)^T\}]$$
$$= \mathrm{tr}\{Var(X)\} + \mathrm{tr}\{Var(Z\hat{\beta})\} = \sigma^2 n - \sigma^2 \mathrm{tr}\{(Z^T Z)^- Z^T Z\}$$
$$= \sigma^2 [n - \mathrm{tr}\{(Z^T Z)^- Z^T Z\}]$$

<u>2.Answer</u>

$P_0(Y = 5) = 1/243 = 0.004$

$P_0(Y = 4) = 10/243 = 0.041$

$P_0(Y = 3) = 40/243 = 0.165$

Solving $P_0(Y = 5) + P_0(Y = 4) + \gamma P_0(Y = 3) = 0.05$,

$$\gamma = \frac{0.05 - 0.004 - 0.041}{0.165} = 0.030.$$

$$T(Y)=\begin{cases}1 & Y>3\\ 0.03 & Y=3\\ 0 & Y<3\end{cases}$$