Homework#3, Statistical Inference I, 2011 Fall

1. Skewness parameter

The gamma distribution has a density

$$f(x) = \frac{1}{\Gamma(\alpha)\beta^{\alpha}} x^{\alpha-1} e^{-x/\beta} I(x > 0)$$

w.r.t. a Lebesgue measure. We observe iid samples $(X_1, ..., X_n)$ having the density f .

1) Find the cumulant generating function $\kappa(t) = \log E[\exp(tX_1)]$

2) Let X be an arbitrary random variable having the m.g.f. in the neighborhood of zero. Show that $\kappa'(0) = E(X)$, $\kappa''(0) = E(X - E[X])^2$, $\kappa'''(0) = E(X - E[X])^3$. (you can use properties of the m.g.f.)

3) For the Gamma distribution, find the skewness parameter $\theta = \frac{\mu_3}{(\mu_2)^{3/2}}$ where $\mu_n = E(X - E[X])^n$ is the central moment.

4) Find a consistent estimator of θ, which is an explicit function of $\overline{X}$ and $S^2 = 1/(n-1)\sum_{i=1}^{n}(X_i - \overline{X})$ (answer if not unique). Prove the consistency.

2. From Exercise 63

Let $(X_1, ..., X_n)$ be iid from $N(\mu, \sigma^2)$. Consider the estimation of σ^2 with the squared error loss.

(a) Show that $(n-1)S^2 / n$ is better than $S^2 = 1/(n-1)\sum_{i=1}^{n}(X_i - \overline{X})$.

(b) Can you find the best estimator of the form cS^2, where c>0 is a constant ?

3. Give a detailed proof of Theorem 3.2 (i)

Answer 1,

1) $M(t) = \dfrac{1}{(1-\beta t)^{\alpha}}$, $\kappa(t) = -\alpha \log(1-\beta t)$.

3) $\kappa'(t) = \dfrac{\alpha\beta}{1-\beta t}$, $\kappa''(t) = \dfrac{\alpha\beta^2}{(1-\beta t)^2}$, $\kappa'''(t) = \dfrac{2\alpha\beta^3}{(1-\beta t)^3}$,

$\mu_2 = \alpha\beta^2$, $\mu_3 = 2\alpha\beta^3$, $\alpha_3 = \dfrac{\mu_3}{(\mu_2)^{3/2}} = \dfrac{2}{\sqrt{\alpha}}$.

4): A consistent estimator of $2/\sqrt{\alpha}$ is $2S/\bar{X}$ (answer is not unique).

$\because$ Since $E(X) = \alpha\beta$ and $Var(X) = \alpha\beta^2$, it follows that

$$\frac{2}{\sqrt{\alpha}} = \frac{\sqrt{Var(X)}}{E(X)}.$$

Since, $\bar{X} \xrightarrow{P} E(X)$ and $S^2 = 1/(n-1)\sum_{i=1}^{n}(X_i - \bar{X}) \xrightarrow{P} Var(X)$, by Slutsky's theorem, $\dfrac{2\sqrt{S^2}}{\bar{X}} \xrightarrow{P} \dfrac{2}{\sqrt{\alpha}}$.

<u>Derivation based on the moment equations:</u>

Since $E(X) = \alpha\beta$, and $Var(X) = \alpha\beta^2$, we set $\bar{X} = \alpha\beta$ and $S^2 = \alpha\beta^2$. Hence, $\hat{\alpha} = \bar{X}^2 / S^2 \to_p \alpha$ by Slutsky's theorem. Again, by Slutsky's theorem, $2S/\bar{X} = 2/\sqrt{\hat{\alpha}} \to_p 2/\sqrt{\alpha}$.