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Midterm Exam, Quality control, 2018 Spring [+ 30 points]

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Q1. [+5] Let $X \sim Bin(n=22, p=0.5)$. Answer up to 3 digits 0.xxx

(1) [+1] Approximate $\Pr(X \le 10)$ by the normal approximation with continuity correction.

$X \sim Bin(n=22, p=0.5)$, $np = 11$, $np(1-p) = 5.5$

$$P(X \le 10) \approx P\left(\frac{X-11}{\sqrt{5.5}} \le \frac{10.5-11}{\sqrt{5.5}}\right) = P(Z \le -0.213)$$

$$= 1 - \Phi(0.213) \approx 1 - 0.584 = 0.416 \text{ ※}$$

(2) [+1] Approximate $\Pr(X = 10)$ by the normal approximation with continuity correction.

$$P(X=10) \approx P\left(\frac{9.5-11}{\sqrt{5.5}} \le \frac{X-11}{\sqrt{5.5}} \le \frac{10.5-11}{\sqrt{5.5}}\right)$$

$$= P(-0.64 \le Z \le -0.213)$$

$$= \Phi(-0.213) - \Phi(-0.64) = 1 - \Phi(0.213) - [1 - \Phi(0.64)]$$

$$= \Phi(0.64) - \Phi(0.213) \approx 0.739 - 0.584 = 0.155 \text{ ※}$$

(3) [+1] Calculate the exact value $\Pr(X=10)$ using using $\binom{22}{10} = 646646$

$$P(X=10) = \binom{22}{10} 0.5^{10} \cdot 0.5^{12}$$

$$= 646646 \cdot (0.5)^{22} \approx 0.154 \text{ ※}$$

(4) [+1] What are the conditions of the normal approximation for n and p? Are they satisfied?

the conditions of normal approximation is $np \ge 10$ and $0.1 \le p \le 0.9$

$\because$ $np = 22 \times 0.5 = 11 > 10$, $0.1 \le p = 0.5 \le 0.9$. $\therefore$ they are satisfied ※

(5) [+1] If $p = 0.3$, how many samples (n) are necessary to get a good approximation?

$0.1 \le p = 0.3 \le 0.9$

$\because$ $np \ge 10 \Rightarrow n \cdot 0.3 \ge 10 \Rightarrow n \ge 33.333$

$\therefore$ sample size take $n \ge 34$ to get a good approximation ※

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Q2. [+5] Data, $X_{ij}, i=1,\cdots,m, \quad j=1,\cdots,n \overset{iid}{\sim} N(\mu,\sigma^2)$, are collected as follows:

					Mean	Range
i=1	12	16	10	10	12	6
i=2	22	20	22	16	20	6
i=3	15	16	12	11	13.5	5
i=4	15	13	12	12	13	3
i=5	14	14	9	11	12	5

$n=4$
$m=5$

(1) [+1] Calculate $\hat{\sigma}$ by the range method.

$$\bar{R} = \frac{1}{5}(6+6+5+3+5) = 5$$

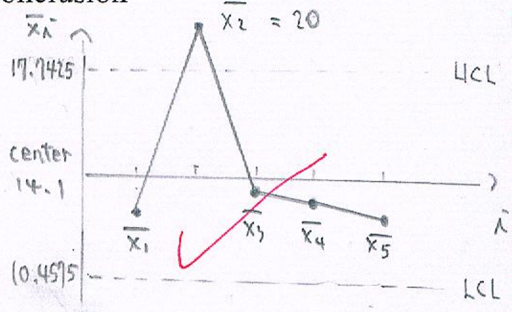
$$\therefore \hat{\sigma} = \frac{\bar{R}}{d_2} = \frac{5}{2.059} \simeq 2.4284$$

(2) [+2] Draw $\overline{X}$-chart (3-sigma quality performance) with your conclusion

$$\text{Center} = \bar{\bar{x}} = \frac{1}{5}(12+20+13.5+13+12) = 14.1$$

$$\text{UCL} = \bar{\bar{x}} + 3\frac{\bar{R}}{d_2\sqrt{n}} = 14.1 + 3\frac{5}{2.059\cdot\sqrt{4}} \simeq 17.7425$$

$$\text{LCL} = 14.1 - 3\frac{5}{2.059\cdot\sqrt{4}} \simeq 10.4575$$



$\because \bar{x}_2 = 20 > \text{UCL}$

$\therefore$ the process is out-of-control

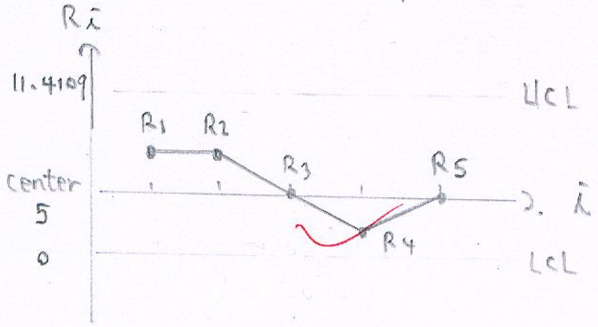
(3) [+2] Draw R-chart (3-sigma quality performance) with your conclusion

$$\bar{R} = 5$$

$$\text{center} = \bar{R} = 5$$

$$\text{UCL} = \bar{R} + 3\frac{d_3}{d_2}\bar{R} = 5 + 3\frac{0.88}{2.059}\cdot 5 \simeq 11.4109$$

$$\text{LCL} = 5 - 3\frac{0.88}{2.059}\cdot 5 \simeq -1.4108 \quad \text{take LCL} = 0 \quad \because \text{Range is positive}$$



$\because$ all point are in (LCL, UCL)

$\therefore$ the process is in control

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Q3 [+6] Let $X_{ij},\ i=1,\ldots,m,\ \ j=1,\ldots,n \overset{iid}{\sim} N(\mu,\sigma^2)$. Consider $\bar{X}$-chart under the target $\mu=\mu_0$ with 3-sigma control limits. Suppose σ^2 is known.

(1) [+2] Under $\mu_1=\mu_0+k\sigma$, derive the consumer's risk (Type II error) in terms of k and Φ.

$$\beta = P\left(\left|\frac{X-\mu_0}{\sigma/\sqrt{n}}\right| \le 3 \,\middle|\, \mu \neq \mu_0\right) = P\left(-3 \le \frac{X-\mu_1+\mu_1-\mu_0}{\sigma/\sqrt{n}} \le 3 \,\middle|\, \mu \neq \mu_0\right)$$

$$= P\left(-3 - \frac{\sqrt{n}(\mu_1-\mu_0)}{\sigma} \le Z \le 3 - \frac{\sqrt{n}(\mu_1-\mu_0)}{\sigma}\right) \quad \text{where } \mu_1 = \mu_0 + k\sigma.$$

$$= \Phi(3-k\sqrt{n}) - \Phi(-3-k\sqrt{n}) \quad *$$

(2) [+2] Draw the OC curve for $n = 4$ (the curve must be clear and detailed)

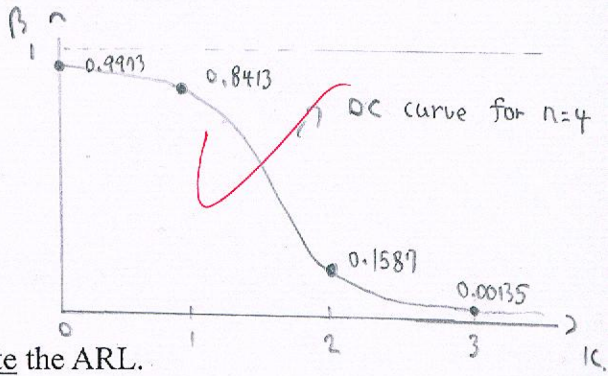
$n=4 \Rightarrow \beta = \Phi(3-2k) - \Phi(-3-2k)$

$k=0 \Rightarrow \beta = \Phi(3) - \Phi(-3) \approx 0.9973$

$k=1 \Rightarrow \beta = \Phi(1) - \underbrace{\Phi(-5)}_{\approx 0} \approx \Phi(1) \approx 0.8413$

$k=2 \Rightarrow \beta \approx \Phi(-1) = 1-\Phi(1) = 0.1587$

$k=3 \Rightarrow \beta \approx \Phi(-3) = 1-\Phi(3) = 0.00135$



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(3) [+2] Under $\mu_1=\mu_0+\sigma$ and $n = 4$, derive and calculate the ARL.

$\beta = \Phi(3-k\sqrt{n}) - \Phi(-3-k\sqrt{n})$, where $n=4$, $k=1$.

$\Rightarrow \beta = \Phi(1) - \Phi(-5) \approx \Phi(1) - 0 = 0.8413$

under $\mu_1 = \mu_0 + \sigma$, $L = \min\{ i \mid \bar{x}_i > UCL \text{ or } \bar{x}_i < LCL \}$

$$ARL_1 = E(L) = E(L \mid \bar{x}_1 \in (LCL, UCL]) \, P(\bar{x}_1 \in (LCL, UCL) \mid \mu_1)$$

$$+ E(L \mid \bar{x}_1 \notin [LCL, UCL]) \, P(\bar{x}_1 \notin (LCL, UCL) \mid \mu_1)$$

$$= [1 + E(L)]\beta + 1 \cdot (1-\beta)$$

$$\Rightarrow E(L) - \beta E(L) = 1 \Rightarrow E(L) = \frac{1}{1-\beta} = ARL_1$$

$$\therefore ARL_1 = \frac{1}{1-0.8413} \approx 6.3012 \quad *$$

↓

under $\mu_1 = \mu_0 + \sigma$

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Q4 [+6] Consider $\overline{X}$-chart with $\mu = \mu_0$ being the target value and σ being known.

(1) [+2] An engineer sets a 3σ-chart with $\mathrm{ARL}_1 \le 2$ under $\mu_1 = \mu_0 + \sigma$. What is the required sample size n?

$$\mathrm{ARL}_1 = \frac{1}{1-\beta} \le 2 \Rightarrow 1 \le 2 - 2\beta \Rightarrow \beta \le 0.5.$$

where $L = 3,\ k = 1$

$$\therefore \beta = \Phi(3 - 1\cdot\sqrt{n}) - \Phi(-3 - 1\cdot\sqrt{n}) \approx \Phi(3 - \sqrt{n}) - 0$$

$$= \Phi(3 - \sqrt{n}) \le 0.5 \Rightarrow 3 - \sqrt{n} \le \Phi^{-1}(0.5) = 0 \Rightarrow n \ge 9$$

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(2) [+2] Draw the OC curve for n obtained from the previous question (show details).

$$\beta = \Phi(3 - k\sqrt{9}) - \Phi(-3 - k\sqrt{9})$$

$$= \Phi(3 - 3k) - \Phi(-3 - 3k)$$

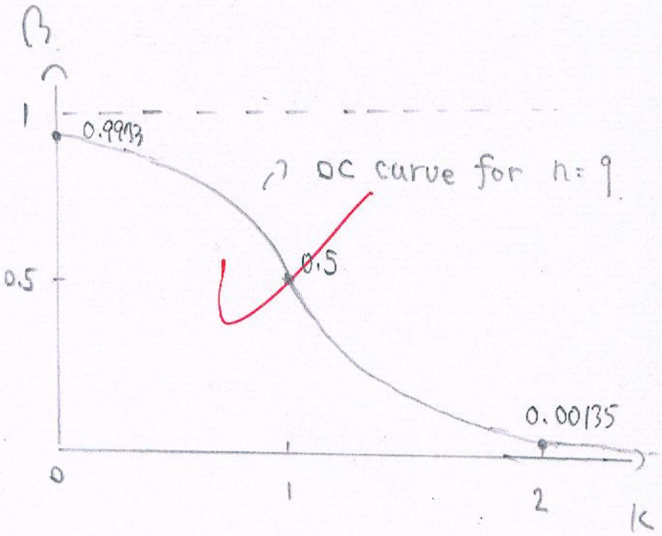
$k = 0 \Rightarrow \beta = \Phi(3) - \Phi(-3) \approx 0.9973$

$k = 1 \Rightarrow \beta = \Phi(0) - \Phi(-6)$

$\approx \Phi(0) - 0 = 0.5$

$k = 2 \Rightarrow \beta \approx \Phi(-3) = 1 - \Phi(3) = 1 - 0.99865$

$= 0.00135.$



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(3) [+2] Calculate the sample autocorrelation at lag k=1 for the following data.

X_t	4	4	3	7	6	8	8	7	7	6
$X_t - \overline{X}_t$	-2	-2	-3	1	0	2	2	1	1	0

$\overline{X}_t = 6.$

$$\sum_{t=1}^{10} (X_t - \overline{X}_t)^2 = (-2)^2 + (-2)^2 + \dots + 1^2 + 0^2 = 28.$$

$$\sum_{t=2}^{10} (X_t - \overline{X}_t)(X_{t-1} - \overline{X}_t) = (-2)(-2) + (-2)(-3) + \dots + 1\cdot 0 = 14.$$

$$\therefore r_1 = \frac{14}{28} = 0.5$$

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Q5 [+8] Consider a p-chart with the target value p_0 and 3-sigma limits.

(1) [+1] Derive consumer's risk under $p \neq p_0$ in terms of binomial probabilities. $D \sim Bin(n, p)$

$$\beta = P(LCL \le \hat{p} \le UCL \mid p \neq p_0) = P(np_0 - 3\sqrt{np_0(1-p_0)} \le D \le np_0 + 3\sqrt{np_0(1-p_0)} \mid p \neq p_0)$$

$$= P(D \le np_0 + 3\sqrt{np_0(1-p_0)}) - P(D \le np_0 - 3\sqrt{np_0(1-p_0)})$$

$$= \sum_{x=0}^{[np_0+3\sqrt{np_0(1-p_0)}]} \binom{n}{x} p^x (1-p)^{n-x} - \sum_{x=0}^{[np_0-3\sqrt{np_0(1-p_0)}]} \binom{n}{x} p^x (1-p)^{n-x}$$

(2) [+2] Calculate <u>consumer's risk</u> and <u>ARL</u> under $p_0 = 0.01$, $p = 0.04$ and $n = 50$.

$np_0 + 3\sqrt{np_0(1-p_0)} \fallingdotseq 2.610$, $np_0 - 3\sqrt{np_0(1-p_0)} \fallingdotseq -1.610$

$$\beta = \sum_{x=0}^{[2.610]} \binom{50}{x} 0.04^x (0.96)^{50-x} - \sum_{x=0}^{[-1.610]} \binom{50}{x} 0.04^x (0.96)^{50-x}$$

$$\fallingdotseq \sum_{x=0}^{2} \binom{50}{x} 0.04^x (0.96)^{50-x} - 0 \fallingdotseq 0.676714$$

$\therefore ARL = \dfrac{1}{1-\beta} = \dfrac{1}{1-0.676714} \fallingdotseq 3.094$

(3) [+2] Calculate <u>producer's risk</u> and <u>ARL</u> under $p_0 = 0.01$ and $n = 50$.

$$\alpha = P(\hat{p} > UCL \text{ or } \hat{p} < LCL \mid p = p_0) = P(D > np_0 + 3\sqrt{np_0(1-p_0)} \mid p_0) + P(D < np_0 - 3\sqrt{np_0(1-p_0)} \mid p_0)$$

$$= 1 - \sum_{x=0}^{[2.610]} \binom{50}{x} (0.01)^x (0.99)^{50-x} - \sum_{x=0}^{[-1.610]} \binom{50}{x} (0.01)^x (0.99)^{50-x}$$

$$\fallingdotseq 1 - \sum_{x=0}^{2} \binom{50}{x} (0.01)^x (0.99)^{50-x} - 0 = 1 - 0.9861827 \fallingdotseq 0.0138173$$

$\therefore ARL = \dfrac{1}{\alpha} = \dfrac{1}{0.0138173} \fallingdotseq 72.373$

(4) [+3] An engineer conducts a waterproof testing for 5 electric boards. The number of defective circuits on each board is:

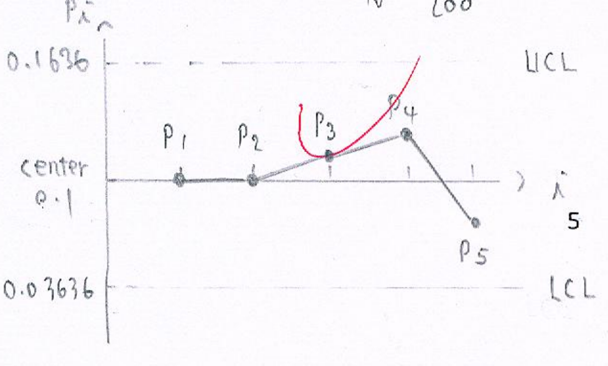
Board ID	1	2	3	4	5
The number of defectives circuits	20	20	23	25	12
The number of circuits	200	200	200	200	200
Defect rates	0.1	0.1	0.115	0.125	0.06

Draw a p-control chart and state your conclusion.

$$\bar{p} = \frac{20+20+23+25+12}{5 \times 200} = \frac{100}{5 \times 200} = 0.1$$

$$UCL = \bar{p} + 3\sqrt{\frac{\bar{p}(1-\bar{p})}{n}} = 0.1 + 3\sqrt{\frac{0.1 \times 0.9}{200}} \fallingdotseq 0.1636$$

$$LCL = 0.1 - 3\sqrt{\frac{0.1 \times 0.9}{200}} \fallingdotseq 0.03636$$



all point are in (LCL, UCL)

$\therefore$ the process is in control

Tables

Table: the c.d.f. of N(0,1) $\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx$

z	0	0.213	0.5	0.64	1	1.5	1.96	2	2.5	3	3.5	4
P	0.5	0.584	0.6915	0.739	0.8413	0.9332	0.975	0.9772	0.9938	0.99865	0.99977	0.99997

Table: Conversion of range to standard deviation

n	2	3	4	5	6
d_2	1.128	1.683	2.059	2.326	2.534
d_3	0.853	0.888	0.880	0.864	0.848

Binomial probabilities.

```
> pbinom(0:5,size=50,prob=0.04)
[1] 0.1298858 0.4004812 0.6767140 0.8608692 0.9510285 0.9855896
> pbinom(0:5,size=50,prob=0.01)
[1] 0.6050061 0.9105647 0.9861827 0.9984038 0.9998543 0.9999891
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