Final Exam, Quality control 2018, Fall [+ 40 points]

37 best

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- **Not only answer but also calculation**

+6 Q1 [+6] The following data are obtained m=20 times from $n = 150$ titanium forgings :

Time (day)	The number of nonconformings	Time (day)	The number of nonconformings
1	8	11	6
2	1	12	0
3	3	13	4
4	0	14	0
5	2	15	3
6	4	16	1
7	0	17	15
8	1	18	2
9	10	19	3
10	6	20	0

+1 (1) [+1] Estimate the fraction nonconforming p.

$$\hat{p} = \frac{1}{mn}\sum_{i=1}^{20} x_i = \frac{1}{150 \cdot 20} \cdot 69 = 0.023.$$

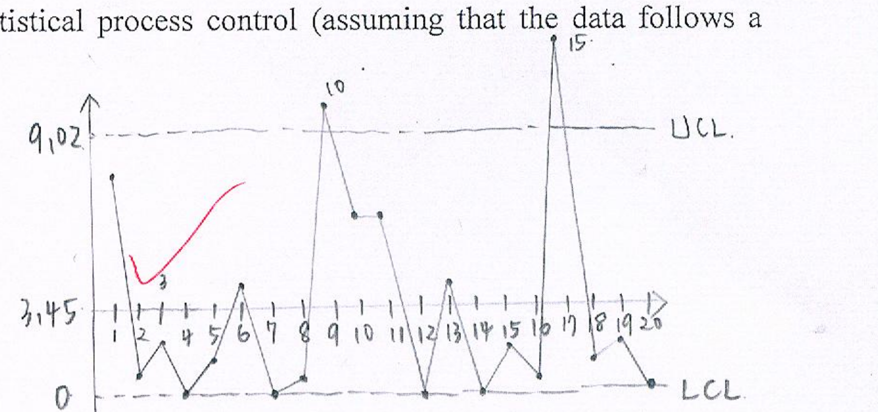
+3 (2) [+3] Draw a c-chart and perform statistical process control (assuming that the data follows a Poisson distribution)

center: $c = \frac{1}{20}\sum_{i=1}^{20} x_i = \frac{1}{20} \cdot 69 = 3.45$

$UCL = c + 3\sqrt{c} = 3.45 + 3\sqrt{3.45} = 9.02$

$LCL = c - 3\sqrt{c} = 3.45 - 3\sqrt{3.45} = -2.12.$

tag $LCL = 0$, since x is always greater then 0.,



- process is out-of-control, since $X_9 = 10 > UCL$, $X_{17} = 15 > UCL$.

+1 (3) [+1] Check the conditions of the Poisson approximation to the Binomial.

$$\begin{cases} \hat{p} < 0.1 \\ n \text{ is large} \end{cases}$$

$\because \hat{p} = 0.023 < 0.1$

$\therefore$ use poisson approximation is better,

+1 (4) [+1] Check the conditions of the normal approximation to the Binomial.

$$\begin{cases} 0.1 \le \hat{p} \le 0.9 \\ np > 10 \end{cases}$$

$\because np = 150 \cdot 0.023 = 3.45 < 10$

$\therefore$ use normal approximation is not good.

+10

Q2 [+10] Let $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, \dots, m$, where σ is known. Let $H_0: \mu = \mu_0$ vs. $H_1: \mu = \mu_1^{as}$, where μ_0 is the target value, $\mu_1^{as} \neq \mu_0$. Let $K = |\mu_1^{as} - \mu_0|/2$ be a reference value.

+2 (1) [+2] Derive $X_i - (\mu_0 + K) > 0$ from a likelihood ratio test.

$$\frac{f(x_i \mid \mu_1^{as})}{f(x_i \mid \mu_0)} = \frac{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(X_i - \mu_1^{as})^2\right)}{\frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{1}{2\sigma^2}(X_i - \mu_0)^2\right)} = \exp\left(-\frac{1}{2\sigma^2}\left[(X_i - \mu_0 + \mu_0 - \mu_1^{as})^2 - (X_i - \mu_0)^2\right]\right)$$

tag ln

$$= \exp\left(-\frac{1}{2\sigma^2}\left[2(X_i - \mu_0)(-2k) + 4k^2\right]\right) > 1 \Leftrightarrow \frac{2k}{\sigma^2}((X_i - \mu_0) - k) > 0$$

$$\Leftrightarrow (X_i - (\mu_0 + k)) > 0 \quad \text{for } \mu_1^{as} > \mu_0$$

+1 (2) [+1] Let $C_1^+ = \max(0, X_1 - (\mu_0 + K))$ be a positive out-of-control signal for $i = 1$.

Define a CUSUM C_i^+ for $i = 2, \dots, m$. $\quad C_i^+ = \max(0, X_i - (\mu_0 + k) + C_{i-1}^+)$ for $i = 2, \dots, m$

+2 (3) [+2] Derive $(\mu_0 - K) - X_i > 0$ from a likelihood ratio test.

$$\frac{f(x_i \mid \mu_1^{as})}{f(x_i \mid \mu_0)} = \exp\left(-\frac{1}{2\sigma^2}\left(2(X_i - \mu_0)(\mu_0 - \mu_1^{as}) + (\mu_0 - \mu_1^{as})^2\right)\right)$$

$$= \exp\left(-\frac{1}{2\sigma^2}\left(2(X_i - \mu_0)2k + 4k^2\right)\right) > 1$$

$$\Leftrightarrow -\frac{4k}{2\sigma^2}((X_i - \mu_0) + k) > 0 \Leftrightarrow -(X_i - \mu_0) - k > 0 \Leftrightarrow (\mu_0 - k) - X_i > 0 \quad \text{for } \mu_0 > \mu_1^{as}$$

+1 (4) [+1] Let $C_1^- = \max(0, (\mu_0 - K) - X_1)$ be a negative out-of-control signal for $i = 1$.

Define a CUSUM C_i^- for $i = 2, \dots, m$. $\quad C_i^- = \max(0, (\mu_0 - k) - X_i + C_{i-1}^-)$ for $i = 2, \dots, m$

+1 (5) [+1] Define the run length in terms of C_i^+ and C_i^- for a decision interval $H = h\sigma$.

run length : $\min\{i : C_i^+ > H \text{ or } C_i^- > H\}$

+1 (6) [+1] Write the ARL in terms of ARL$^+$ and ARL$^-$ for C_i^+ and C_i^- (including derivations).

$$ARL = \frac{1}{P(C_i^+ < H \text{ or } C_i^- < H)} = \frac{1}{P(C_i^+ < H) + P(C_i^- < H)}$$

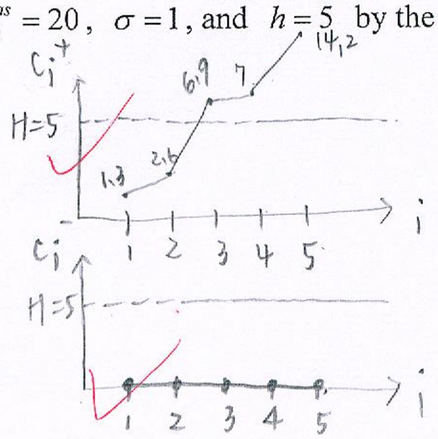
$$ARL^+ = \frac{1}{P(C_i^+ < H)} \Rightarrow P(C_i^+ < H) = \frac{1}{ARL^+} \quad \Rightarrow ARL = \frac{1}{\frac{1}{ARL^+} + \frac{1}{ARL^-}} \Rightarrow \frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}$$

+2 (7) [+2] Perform a process control under $\mu_0 = 10$, $\mu_0^{as} = 20$, $\sigma = 1$, and $h = 5$ by the CUSUM:

X_i	16.3	16.3	19.3	15.1	22.2

$k = 5$

X_i	$X_i - (\mu_0 + k)$	C_i^+	$(\mu_0 - k) - X_i$	C_i^-
16.3	1.3	1.3	-11.3	0
16.3	1.3	2.6	-11.3	0
19.3	4.3	6.9	-14.3	0
15.1	0.1	7	-10.1	0
22.2	7.2	14.2	-17.2	0



process is out-of control, since $C_i^+ > H$ for $i = 3, 4, 5$

+8

Q3 [+8] We consider the binomial CUSUM under $X_i \sim Bin(n,p)$, $i=1,\dots,m$.

+2 (1) [+2] Derive $X_i - nk > 0$ for some k from a likelihood ratio test for

$$H_1 : p = p_0 \quad \text{vs.} \quad H_1 : p = p_1^{as} > p_0 .$$

$$\frac{f(x_i|p_1^{as})}{f(x_i|p_0)} = \frac{\binom{n}{x_i} p_1^{as\,x_i}(1-p_1^{as})^{n-x_i}}{\binom{n}{x_i} p_0^{x_i}(1-p_0)^{n-x_i}} = \left(\frac{p_1^{as}(1-p_0)}{p_0(1-p_1^{as})}\right)^{x_i}\left(\frac{1-p_1^{as}}{1-p_0}\right)^{n} > 1$$

$$\left(\frac{f(x_i|p_1^{as})}{f(x_i|p_0)} > 1 \Leftrightarrow \ln\frac{f(x_i|p_1^{as})}{f(x_i|p_0)} > 0.\right)$$

$$\Leftrightarrow x_i \ln\frac{p_1^{as}(1-p_0)}{p_0(1-p_1^{as})} + n\ln\frac{1-p_1^{as}}{1-p_0} > 0$$

$$\Leftrightarrow x_i + n\underbrace{\frac{\ln\left(\frac{1-p_1^{as}}{1-p_0}\right)}{\ln\left(\frac{p^{as}(1-p_0)}{p_0(1-p_1^{as})}\right)}}_{=-k} > 0$$

$\Leftrightarrow X_i - nk > 0$ for $p_1^{as} > p_0$.

∦

+1 (2) [+1] Compute k under $p_0 = 1/4$ and $p_1^{as} = 1/2$.

$$-k = \frac{\ln\left(\frac{1-p_1^{as}}{1-p_0}\right)}{\ln\left(\frac{p^{as}(1-p_0)}{p_0(1-p_1^{as})}\right)} = \frac{\ln\left(\frac{0,5}{0,75}\right)}{\ln\left(\frac{0,5\cdot 0,75}{0,25\cdot 0,5}\right)} \fallingdotseq -0,37 \Leftrightarrow k \fallingdotseq 0,37.$$ ∦

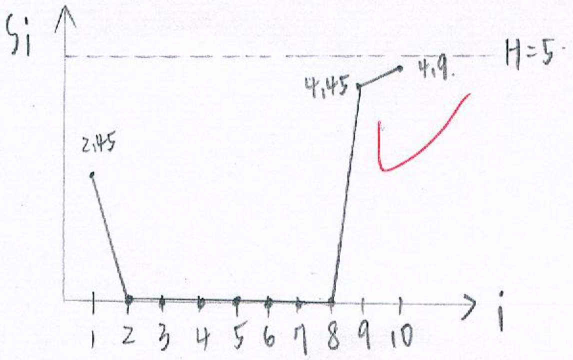
+2 (3) [+2] Use the above k to fill in the table for the binomial CUSUM to detect a positive shift:

$S_0 = 0$, $\quad S_i = \max\{0, X_i - nk + S_{i-1}\}$, $\quad i = 1, 2, \dots 10$, $\quad n = 15$

$nk = 15 \times 0,37 = 5,55$

i	X_i	$X_i - nk$ (5,55)	S_i
1	8	2,45	2,45
2	1	-4,55	0
3	3	-2,55	0
4	0	-5,55	0
5	2	-3,55	0
6	4	-1,55	0
7	0	-5,55	0
8	1	-4,55	0
9	10	4,45	4,45
10	6	0,45	4,9

+2 (4) [+2] Perform process control by using the CUSUM with a decision interval $H = 5$.



· Process is in-control since all of $S_i < H$ for $i = 1, 2, \dots, 10$.

+1 (5) [+1] Find the change point estimator $\hat{\tau}_{CUSUM} = \max\{i : S_i = 0\}$.

$\hat{\tau} = 8$

∦

+9

Q4 [+10] Consider data $\begin{pmatrix} X_{ij1} \\ X_{ij2} \end{pmatrix} \sim N\left[\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}\right]$, $i = 1,\ldots,m$, $j = 1,\ldots,16$.

+2 (1) [+3] Define a univariate $\bar{X}$-chart for $\mu_1 = 1$ (target value).

Define

1. ~~X~~Plot $\bar{X}_i$ for $i=1,2,\ldots,m$. $\sqrt{5}$

2. Center= $\mu_1=1$, UCL= $\cancel{\mu+}3\frac{5}{\sqrt{16}}=4.75$, LCL= $\cancel{\mu-}3\frac{5}{\sqrt{16}}=-2.75$.

3. Process is out-of-control if $\bar{X}_i > UCL$ or $\bar{X}_i < LCL$ for $i=1,2,\ldots,m$.

+1 (2) [+1] Derive the eigenvalues of $\Sigma = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$

$$\begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 16 = \lambda^2 - 10\lambda + 9 = (\lambda-1)(\lambda-9) = 0$$

$$\Rightarrow \lambda_1 = 9,\ \lambda_2 = 1.$$

+1 (3) [+1] Derive the eigenvectors for each eigenvalue.

for $\lambda_1 = 9$ $\begin{pmatrix} -4 & -4 \\ -4 & -4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow -4x_1 - 4x_2 = 0 \Rightarrow x_1 = -x_2$, eigenvector $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = e_1$

for $\lambda_1 = 1$ $\begin{pmatrix} 4 & -4 \\ -4 & 4 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Leftrightarrow 4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$, eigenvector $= \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = e_2$.

+5 (4) [+5] Draw a control ellipse for monitoring $\boldsymbol{\mu} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ under $\alpha = 0.010$. Use $\chi^2_{\alpha=0.010, df=2} = 9.21$

$$\sigma_1^2\sigma_2^2 - \sigma_{12}^2 = 25 - 16 = 9.$$

$$\chi_0^2 = \frac{16}{9}\{5(\bar{x}_1-1)^2 - 8(\bar{x}_1-1)(\bar{x}_2-1) + 5(\bar{x}_2-1)^2\}.$$

for $\bar{x}_2 = \mu_2 = 1$

$$\frac{16}{9}\cdot 5(\bar{x}_1-1)^2 \le 9.21$$

$$(\bar{x}_1-1)^2 \le 9.21 \cdot \frac{9}{16} \cdot \frac{1}{5} = 1.036.$$

$$1 - 1.018 \le \bar{x}_1 \le 1 + 1.018$$

$$-0.018 \le \bar{x}_1 \le 2.018$$

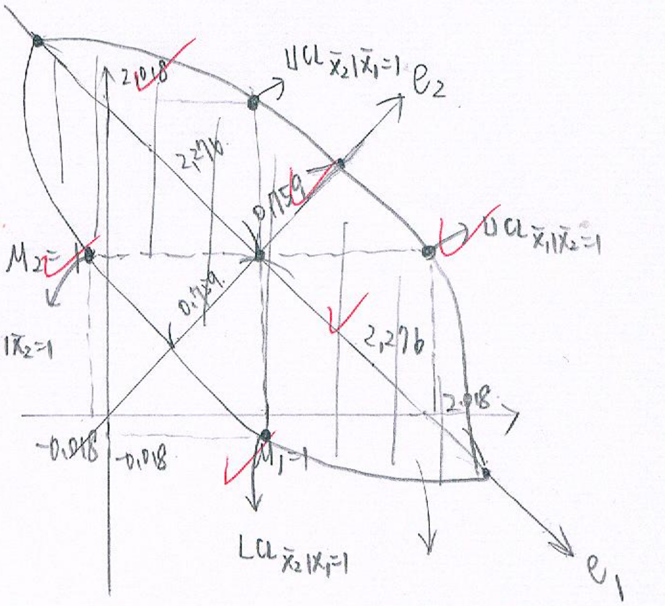
for $\bar{x}_1 = \mu_1 = 0$

$$\frac{16}{9}\cdot 5(\bar{x}_2-1)^2 \le 9.21$$

$$\Rightarrow -0.018 \le \bar{x}_2 \le 2.018$$

$$\sqrt{\lambda_1 \frac{\chi^2_{0.01,2}}{16}} = \sqrt{9 \cdot \frac{9.21}{16}} = 2.276$$

$$\sqrt{\lambda_2 \frac{\chi^2_{0.01,2}}{16}} = \sqrt{1 \cdot \frac{9.21}{16}} = 0.759$$



joint control region

+4

Q5 [+6] Let $\begin{pmatrix} X_{ij1} \\ X_{ij2} \end{pmatrix} \sim N\left[\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}\right]$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

+1 (1) [+1] Define unbiased estimators of μ_1 and μ_2, respectively:

$$\bar{\bar{X}}_1 = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_{i1}, \quad \bar{\bar{X}}_2 = \frac{1}{m}\sum_{i=1}^{m} \bar{X}_{i2}$$

+1 (2) [+2] Define unbiased estimators of σ_1^2, σ_2^2, and σ_{12}, respectively:

$$s_1^2 = \frac{1}{m}\sum_{i=1}^{m} s_{i1}^2, \quad s_{i1}^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_{ij1} - \bar{X}_{i1})^2$$

$$s_2^2 = \frac{1}{m}\sum_{i=1}^{m} s_{i2}^2, \quad s_{i2}^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_{ij2} - \bar{X}_{i2})^2$$

$$s_{12} = \frac{1}{m}\sum_{i=1}^{m} s_{i12}, \quad s_{12} = \sqrt{\frac{\sum(X_{ij1} - \bar{X}_{i1})(X_{ij2} - \bar{X}_{i2})}{\sum(X_{ij1} - \bar{X}_{i1})^2 \sum(X_{ij2} - \bar{X}_{i2})^2}}$$

+1 (3) [+1] Write down T_i^2 statistic (not using a vector or matrix).

$$T_i^2 = \frac{n}{s_1^2 s_2^2 - s_{12}^2}\left(s_2^2(\bar{X}_{i1} - \bar{\bar{X}}_1)^2 - 2s_{12}(\bar{X}_{i1} - \bar{\bar{X}}_1)(\bar{X}_{i2} - \bar{\bar{X}}_2) + s_1^2(\bar{X}_{i2} - \bar{\bar{X}}_1)^2\right)$$

+1 (4) [+1] Define a level α joint control region for $\boldsymbol{\mu} = \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}$ under $T_i^2 \sim \frac{p(m-1)(n-1)}{mn-m-p+1} F_{p,mn-m-p+1}$.

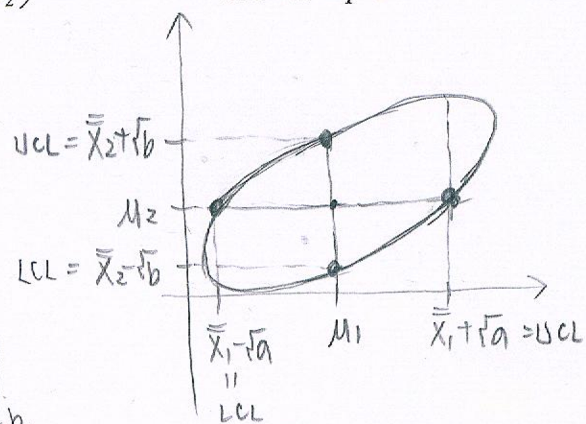
for $\bar{X}_{i2} = \mu_2$

$$\frac{n s_2^2}{s_1^2 s_2^2 - s_{12}^2}(\bar{X}_{i1} - \bar{\bar{X}}_1)^2 < \frac{p(m-1)(n-1)}{mn-m-p+1} F_{p,mn-m-p+1}$$

$$\Rightarrow \bar{X}_{i1} \in \Big(\bar{\bar{X}}_1 \pm \underbrace{\sqrt{\frac{s_1^2 s_2^2 - s_{12}^2}{n s_2^2} \cdot \frac{p(m-1)(n-1)}{mn-m-p+1} \cdot F_{p,mn-m-p+1}}}_{=a}\Big)$$

for $\bar{X}_{i1} = \mu_1$

$$\Rightarrow \bar{X}_{i2} \in \Big(\bar{\bar{X}}_2 \pm \underbrace{\sqrt{\frac{s_1^2 s_2^2 - s_{12}^2}{n s_1^2} \cdot \frac{p(m-1)(n-1)}{mn-m-p+1} F_{p,mn-m-p+1}}}_{=b}\Big)$$



+0 (5) [+1] Show the Hotelling T^2-chart reduces to the χ^2-chart under some condition.

$$T_i^2 \sim \frac{p(m-1)(n-1)}{mn-m-p+1} F_{p,mn-m-p+1}$$

prove. (-1)

as $m \to \infty$, $\frac{p(m-1)(n-1)}{mn-m-p+1} \xrightarrow{P} p$

$$F_{p,mn-m-p+1} \xrightarrow{P} \frac{\chi_p^2}{p}$$

such that as $m \to \infty$

$$T_i^2 \sim \frac{p(m-1)(n-1)}{mn-m-p+1} F_{p,mn-m-p+1} \xrightarrow{P} p \cdot \frac{\chi_p^2}{p} = \chi_p^2$$

#