Final Exam, Quality control 2017 Spring [+ 36 points] +33

Name: 陳楷凱 Tan Kai Chen.

- **Not only answer but also calculation**

+7 **Q1 [+10]** Let $X_i \sim N(\mu, \sigma^2)$, $i = 1, 2, ..., m$, The tabular CUSUM is defined as

$$C_i^+ = \max(0, X_i - (\mu_0 + K) + C_{i-1}^+), \qquad C_i^- = \max(0, (\mu_0 - K) - X_i + C_{i-1}^-),$$

where $C_0^- = C_0^+ = 0$, $\mu = \mu_0$ (target value), and σ =known.

+1 1) [+1] $K = |\mu_1 - \mu_0| / 2 = k\sigma$ is called reference value, and $H = h\sigma$ is called decision interval.

+1 2) [+1] How the tabular CUSUM chart finds out-of-control signals?

if $C_j^+ > H$ or $C_j^- > H$ $\Rightarrow$ out-of-control

+0 3) [+1] Engineers wish to detect a shift $\mu_1 = \mu_0 + \sigma$ quickly. Then, $k = \frac{1}{2}$ and $h =$ ~~5~~ or 1

+0 4) [+1] Define the average run length (ARL)

× ARL= $\left[\frac{1}{ARL^-} + \frac{1}{ARL^+}\right]^{-1}$

× ARL$^+$= $\frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}$, $\Delta = \delta^* - k$, $\delta^* = \frac{\mu_1 - \mu_0}{\sigma}$, $k = \frac{|\mu_1 - \mu_0|}{2}$, $b = h + 1.166$

× ARL$^-$= $\frac{\exp(-2\Delta b) + 2\Delta b - 1}{2\Delta^2}$, $\Delta = -\delta^* - k$, $\delta^* = \frac{\mu_1 - \mu_0}{\sigma}$, $k = \frac{|\mu_1 - \mu_0|}{2}$, $b = h + 1.166$.

+1 5) [+1] Derive the relationship among ARL, ARL$^+$, and ARL$^-$.

$$ARL = E(L) = \frac{1}{P(C_i^+ > H \text{ or } C_i^- > H)} = \frac{1}{P(C_i^+ > H) + P(C_i^- > H)} = \frac{1}{\frac{1}{ARL^+} + \frac{1}{ARL^-}}$$

$$\Rightarrow \frac{1}{ARL} = \frac{1}{ARL^+} + \frac{1}{ARL^-}$$

+2 6) [+3] ARL $\boxed{\text{increase, decrease, unchange}}$ if k increases. Why?

Because k increasing, such that $C_i^+ > H$ or $C_i^- > H$ easyer occures so ARL=E(L) will decreasing.

ARL $\boxed{\text{increase, decrease, unchange}}$ if h increases. Why?

if h increases, will difficult to detect the out-of-control point, so ARL=E(L) will increasing.

ARL $\boxed{\text{increase, decrease, unchange}}$ if σ increases. Why?

× Because, $C_i^+ > H$ or $C_i^- > H$ is easyer occurs, since σ increasing so ARL=E(L) will decreasing.

+2 7) [+2] Under the recommended value of k, ARL$_0$=168 for h=4 and ARL$_0$=465 for h=5. Use an interpolation to approximate h such that ARL$_0$=370.

$$\frac{465-168}{5-4} = \frac{370-168}{x} \Rightarrow 297x = 202 \Rightarrow x = \frac{202}{297} \approx 0.68$$

$$\therefore h \approx 4 + 0.68 = 4.68$$

+10 Q2 [+10] Consider the 1st- order autoregressive model

$$X_t = \xi + \phi X_{t-1} + \varepsilon_t, \quad t = 1, 2, \ldots, m,$$

where $\varepsilon_t \sim N(0, \sigma^2)$, $t = 1, 2, \ldots, m$, and $0 < |\phi| < 1$.

+1 • [+1] Derive $E[X_t]$

$$E(X_t) = E(\xi + \phi X_{t-1} + \varepsilon_t) = \xi + \phi[\xi + \phi E(X_{t-2})] = \xi + \phi\xi + \phi^2\xi + \cdots = \frac{\xi}{1-\phi} \ \#$$

+1 • [+1] Derive $SD[X_t]$

$$var(X_t) = \phi^2 var(X_{t-1}) + var(\varepsilon_t) + 2\underbrace{cov(\phi X_{t-1}, \varepsilon_t)}_{\to 0 \ \because \varepsilon_t \perp\!\!\!\perp \varepsilon_{t-1}} = \phi^2[\phi^2 var(X_{t-2}) + var(\varepsilon_{t-1})] + \sigma^2$$

$$= \sigma^2 + \phi^2\sigma^2 + \phi^4\sigma^2 + \cdots = \frac{\sigma^2}{1-\phi^2}, \quad \therefore SD(X_t) = \sqrt{\frac{\sigma^2}{1-\phi^2}} \ \#$$

+2 • [+2] Derive $Cov[X_t, X_{t-k}]$

$$cov(X_t, X_{t-k}) = cov(\xi + \phi X_{t-1} + \varepsilon_t, X_{t-k}) = \phi\, cov(X_{t-1}, X_{t-k}) + \underbrace{cov(\varepsilon_t, X_{t-k})}_{\to 0 \ \because \varepsilon_t \perp\!\!\!\perp \varepsilon_{t-k}}$$

$$= \phi^2 cov(X_{t-2}, X_{t-k}) = \cdots = \phi^k var(X_{t-k}) = \phi^k \frac{\sigma^2}{1-\phi^2} \ \#$$

+1 • [+1] Derive the autocorrelation function at lag k

$$\rho_{X_t, X_{t-k}} = \frac{cov(X_t, X_{t-k})}{\sqrt{var(X_t)\, var(X_{t-k})}} = \frac{\phi^k \cdot var(X_t)}{var(X_t)} = \phi^k \ \#$$

+2 • [+2] Calculate the sample autocorrelation at lag k=1

X_t	4	4	3	7	6	8	8	7	7	6
$X_t - \bar{X}$	-2	-2	-3	1	6	2	2	1	1	0
$X_{t-1} - \bar{X}$		-2	-2	-3	1	6	2	2	1	1

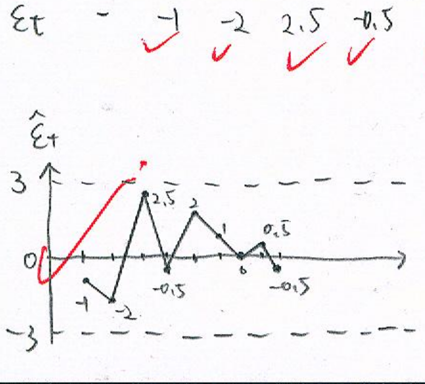
$\bar{X} = 6$

$$r_1 = \frac{4+6+(-3)+6+12+4+2+1+0}{4+4+9+1+36+4+4+1+1+0} = \frac{32}{64} = 0.5$$

+3 • [+3] Draw the residual control chart under $\hat{\xi} = 3$, $\hat{\phi} = 0.5$, and $\hat{\sigma} = 1$.

$\hat{\varepsilon}_t = X_t - \hat{X}_t$, $\quad \hat{X}_t = 3 + 0.5 X_{t-1}$, $t = 2, \cdots$

X_t	4	4	3	7	6	8	8	7	7	6
X_{t-1}	-	4	4	3	7	6	8	8	7	7
$\hat{X}_t$	-	5	5	4.5	6.5	6	7	7	6.5	6.5
$\hat{\varepsilon}_t$	-	-1	-2	2.5	-0.5	2	1	0	0.5	-0.5



The process is in-control since $-3 < \hat{\varepsilon}_t < 3$.

+10

Q3 [+10] Control ellipse

Consider data $\mathbf{X}_1,\dots,\mathbf{X}_{16} \overset{iid}{\sim} N_2(\boldsymbol{\mu},\Sigma)$, where $\boldsymbol{\mu} = \begin{pmatrix}1\\1\end{pmatrix}$, $\Sigma = \begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}$.

+1 (1) [+1] What is the distribution of $\bar{\mathbf{X}} = \frac{1}{16}\sum_{i=1}^{16} \mathbf{X}_i$? $\bar{X} \sim N_2(\underset{\sim}{\mu}, \frac{\Sigma}{16})$ #   write

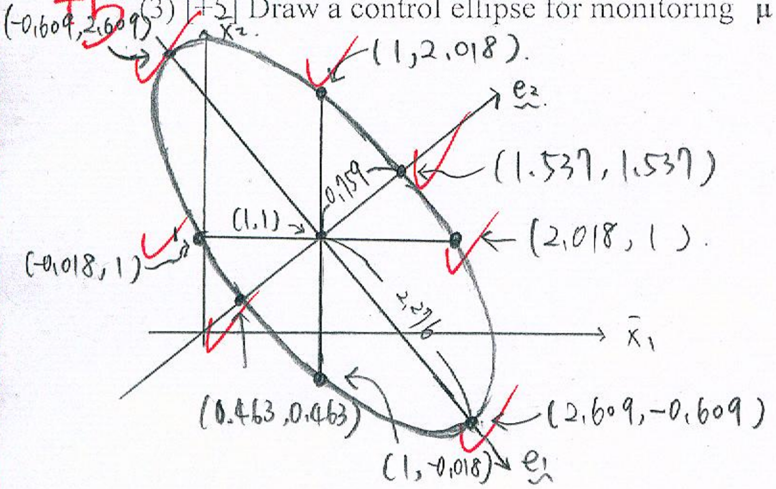
+2 (2) [+2] Derive the eigenvalues and eigenvectors of $\Sigma = \begin{pmatrix}5 & -4\\ -4 & 5\end{pmatrix}$.

$\begin{vmatrix}5-\lambda & -4\\ -4 & 5-\lambda\end{vmatrix} = (5-\lambda)^2-16 = \lambda^2-10\lambda+9 = (\lambda-9)(\lambda-1)$, $\therefore$ $\lambda_1=9$ and $\lambda_2=1$. #

for $\lambda_1=9$, $\begin{bmatrix}-4&-4\\-4&-4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\underset{\sim}{0}$, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\-1\end{bmatrix} \Rightarrow \underset{\sim}{e}_1 = \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix}$ #

for $\lambda_2=1$, $\begin{bmatrix}4&-4\\-4&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\underset{\sim}{0}$, $\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix} \Rightarrow \underset{\sim}{e}_2 = \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$ #

+5 (3) [+5] Draw a control ellipse for monitoring $\boldsymbol{\mu} = \begin{pmatrix}1\\1\end{pmatrix}$ under $\alpha = 0.010$. Use $\chi^2_{\alpha=0.010,df=2} = 9.21$



$$UCL_{\bar{x}_2|\bar{x}_1=1} = \mu_2 + \sqrt{\chi^2_{0.01,2}\,\frac{\sigma_1^2\sigma_2^2-\sigma_{12}^2}{n\cdot\sigma_1^2}} = 1+\sqrt{9.21\,\frac{9}{16\cdot 5}} = 2.018$$

$$LCL_{\bar{x}_2|\bar{x}_1=1} = 1-\sqrt{9.21\,\frac{9}{16\cdot 5}} = -0.018$$

$$UCL_{\bar{x}_1|\bar{x}_2=1} = \mu_1 + \sqrt{\chi^2_{0.01;2}\,\frac{\sigma_1^2\sigma_2^2-\sigma_{12}^2}{n\cdot\sigma_2^2}} = 1+\sqrt{9.21\,\frac{9}{16\cdot 5}} = 2.018$$

$$LCL_{\bar{x}_1|\bar{x}_2=1} = -0.018$$

$$\sqrt{\lambda_1\frac{\chi^2_{0.01;2}}{n}} = \sqrt{9\cdot\frac{9.21}{16}} = 2.276$$

$$\sqrt{\lambda_2\frac{\chi^2_{0.01;2}}{n}} = \sqrt{1\cdot\frac{9.21}{16}} = 0.759$$

$2.276 \times \frac{1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1.609\\-1.609\end{bmatrix}$, $\begin{bmatrix}1.609\\-1.609\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}2.609\\-0.609\end{bmatrix}$

$2.276 \times \frac{-1}{\sqrt2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1.609\\1.609\end{bmatrix}$, $\begin{bmatrix}-1.609\\1.609\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}-0.609\\2.609\end{bmatrix}$

$0.759 \times \frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0.537\\0.537\end{bmatrix}$, $\begin{bmatrix}0.537\\0.537\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}1.537\\1.537\end{bmatrix}$

$0.759 \times \frac{-1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}-0.537\\-0.537\end{bmatrix}$, $\begin{bmatrix}-0.537\\-0.537\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}=\begin{bmatrix}0.463\\0.463\end{bmatrix}$

+2 (4) [+2] Is $\bar{X} = \begin{pmatrix}\bar{X}_1\\ \bar{X}_2\end{pmatrix} = \begin{pmatrix}1/4\\ -1/4\end{pmatrix}$ in-control? Verify your answer.

$$\frac{n}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\left[\sigma_1^2(\bar{x}_2-\mu_2)^2-2\sigma_{12}(\bar{x}_1-\mu_1)(\bar{x}_2-\mu_2)+\sigma_2^2(\bar{x}_1-\mu_1)^2\right]$$

$$=\frac{16}{9}\left[5\cdot\left(-\frac{5}{4}\right)^2+8\cdot\left(-\frac{3}{4}\right)\left(-\frac{5}{4}\right)+5\left(-\frac{3}{4}\right)^2\right] = 32.222 > \chi^2_{0.01,2}=9.21$$

$\therefore$ $\underset{\sim}{\bar{x}} = \begin{pmatrix}1/4\\-1/4\end{pmatrix}$ is out-of-control #

+6

Q4. [+6] χ^2-chart for monitoring μ

The data below are generated from the bivariate normal distribution with

$$N_2\left(\mu = \begin{bmatrix}50\\30\end{bmatrix}, \Sigma = \begin{bmatrix}200 & 100\\100 & 100\end{bmatrix}\right), \ n = 25 \text{ and } m = 15.$$

+4 (1) [+4] Calculate X_0^2 below.

$$X_0^2 = \frac{n}{\sigma_1^2\sigma_2^2-\sigma_{12}^2}\left[\sigma_1^2(\bar{x}_2-\mu_2)^2 - 2\sigma_{12}(\bar{x}_1-\mu_1)(\bar{x}_2-\mu_2) + \sigma_2^2(\bar{x}_1-\mu_1)^2\right]$$

K	$\bar{\mathbf{X}}_k$	X_0^2
1	(58, 32) (50,30)	10
2	(60, 33) (50,30)	14.5
3	(50, 27) (50,30)	4.5
.	.	.
.	.	.
.	.	.
14	(75, 45) (50,30)	81.25
15	(55, 27) (50,30)	18.25

$k=1,\ X_0^2 = \frac{25}{10000}\left[200\cdot(2^2) - 200(8)(2) + 100(8^2)\right] = 10$

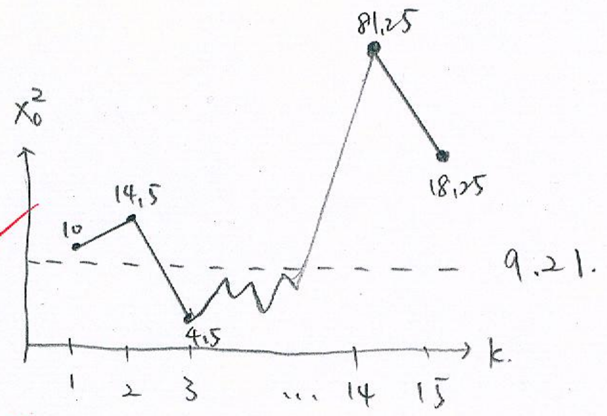
$k=2,\ X_0^2 = \frac{25}{10000}\left[200(3^2) - 200(10)(3) + 100(10^2)\right] = 14.5$

$k=3,\ X_0^2 = \frac{25}{10000}\left[200(-3)^2 - 200(0)(-3) + 100(0)\right] = 4.5$

$k=14,\ X_0^2 = \frac{25}{10000}\left[200(15^2) - 200(25)(15) + 100(25^2)\right] = 81.25$

$k=15,\ X_0^2 = \frac{25}{10000}\left[200(-3)^2 - 200(5)(-3) + 100(5^2)\right] = 18.25$

+2 (2) [+2] Draw χ^2-chart with $\alpha = 0.01$ using $\chi^2_{\alpha=0.010, df=2} = 9.21$.



The process is out-of-control

since $X_{0i}^2 > 9.21,\ i = 1, 2, 14, 15$