Quiz #2, Quality Control, 2013, 4/26(Fri)

1 Profile monitoring [+4] +4/4

An engineer wishes to monitor a linear profile

$$Y_{ij} = 6 + 0.5x_j + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2 = 1),$$

where the design points are $(x_1, x_2, x_3) = (1,2,3)$ with $n = 3$.

1) Find the centered model

2) Group ith data is $(Y_{i1}, x_1) = (5,1), (Y_{i2}, x_2) = (6,2), (Y_{i3}, x_3) = (7,3)$. Fit to the model

$$Y_{ij} = \beta_{i,0} + \beta_{i,1}(x_j - \bar{x}) + \varepsilon_{ij}, \quad \varepsilon_{ij} \sim N(0, \sigma^2),$$

and obtain the LSE $(\hat{\beta}_{i,0}, \hat{\beta}_{i,1})$.

3) Find the marginal distributions of $\hat{\beta}_{i,0}$ and $\hat{\beta}_{i,1}$, respectively, under in-control.

4) Find LCL and UCL for $\hat{\beta}_{i,0}$ and $\hat{\beta}_{i,1}$, respectively, using 3-sigma limit (i.e., $\alpha = 0.0027$). Then, conclude the process is in-control or out-of-control.

(1) $\bar{X} = \frac{1}{3}(1+2+3) = \frac{1}{3} \times 6 = 2$

$Y_{ij} = 6 + 0.5(X_j - \bar{X}) + 0.5\bar{X} + \varepsilon_{ij} \Rightarrow Y_{ij} = 7 + 0.5(X_j - 2) + \varepsilon_{ij}$, $\beta_{i0} = 7$, $\beta_{i1} = 0.5$

(By part *)

(2) $\begin{bmatrix} \hat{\beta}_{i0} \\ \hat{\beta}_{i1} \end{bmatrix} = \begin{bmatrix} \bar{Y}_i \\ \frac{\sum_j Y_{ij}(X_j - \bar{X})}{S_{XX}} \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$ where $\bar{Y}_i = \frac{1}{3}(5+6+7) = 6$

$S_{XX} = \sum_{j=1}^{3} (X_j - \bar{X})^2 = (-1)^2 + 0^2 + 1^2 = 2$

$\sum_{j=1}^{3} Y_{ij}(X_j - \bar{X}) = 5(-1) + 0 + 7 \times 1 = 2$

model: $Y_{ij} = 6 + (X_j - 2) + \varepsilon_{ij}$

(by part *)

(3) $\mathrm{Cov}\begin{bmatrix} \hat{\beta}_{i0} \\ \hat{\beta}_{i1} \end{bmatrix} = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{S_{XX}} \end{bmatrix}$ $\hat{\beta}_{i0} \sim N(7, \frac{1}{3})$, $\hat{\beta}_{i1} \sim N(0.5, \frac{1}{2})$

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(4) chart for $\hat{\beta}_{i0}$: $UCL = \beta_{i0} + 3\sqrt{\frac{\sigma^2}{n}} = 7 + 3\sqrt{\frac{1}{n}} = 8.73$

$LCL = \beta_{i0} - 3\sqrt{\frac{\sigma^2}{n}} = 7 - 3\sqrt{\frac{1}{n}} = 5.27$; $\hat{\beta}_{i0} = 6$ in-control

chart for $\hat{\beta}_{i1}$: $UCL = \beta_{i1} + 3\sqrt{\frac{\sigma^2}{S_{XX}}} = 0.5 + 3\sqrt{\frac{1}{S_{XX}}} = 2.62$

$LCL = \beta_{i1} - 3\sqrt{\frac{\sigma^2}{S_{XX}}} = 0.5 - 3\sqrt{\frac{1}{S_{XX}}} = -1.62$ $\hat{\beta}_{i1} = 1$ in-control.

<part *>.

Let $Y_i = X\beta + \varepsilon_i$; $Y_i = \begin{bmatrix} Y_{i1} \\ \vdots \\ Y_{in} \end{bmatrix}$, $X = \begin{bmatrix} 1 & x_1-\bar{x} \\ \vdots & \vdots \\ 1 & x_n-\bar{x} \end{bmatrix}$, $\beta = \begin{bmatrix} \beta_{i0} \\ \beta_{i1} \end{bmatrix}$, $\varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{in} \end{bmatrix}$

$$\hat{\beta} = \begin{bmatrix} \hat{\beta}_{i0} \\ \hat{\beta}_{i1} \end{bmatrix} = (X'X)^{-1}X'Y_i = \begin{bmatrix} \bar{Y}_i \\ \frac{\sum_j (x_j-\bar{x})Y_{ij}}{S_{xx}} \end{bmatrix}.$$

where

$$(X'X) = \begin{bmatrix} 1 & \cdots & 1 \\ x_1-\bar{x} & & x_n-\bar{x} \end{bmatrix} \begin{bmatrix} 1 & x_1-\bar{x} \\ \vdots & \vdots \\ 1 & x_n-\bar{x} \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & \sum_j (x_j-\bar{x})^2 \end{bmatrix} = \begin{bmatrix} n & 0 \\ 0 & S_{xx} \end{bmatrix}$$

$$(X'X)^{-1} = \frac{1}{nS_{xx}} \begin{bmatrix} S_{xx} & 0 \\ 0 & n \end{bmatrix} = \begin{bmatrix} \frac{1}{n} & 0 \\ 0 & \frac{1}{S_{xx}} \end{bmatrix}$$

$$(X'X)^{-1}X' = \begin{bmatrix} \frac{1}{n} & \cdots & \frac{1}{n} \\ \frac{x_1-\bar{x}}{S_{xx}} & \cdots & \frac{x_n-\bar{x}}{S_{xx}} \end{bmatrix} \quad \therefore \quad (X'X)^{-1}X'y = \begin{bmatrix} \frac{\sum y_j}{n} \\ \frac{\sum_j y_{ij}(x_j-\bar{x})}{S_{xx}} \end{bmatrix}$$

$$\mathrm{cov}\begin{pmatrix} \hat{\beta}_{i0} \\ \hat{\beta}_{i1} \end{pmatrix} = (X'X)^{-1}\sigma^2 = \begin{bmatrix} \frac{\sigma^2}{n} & 0 \\ 0 & \frac{\sigma^2}{S_{xx}} \end{bmatrix}.$$

$\hat{\beta}_{i0} \sim$ Normal , mean $= \beta_{i0}$. variance $= \frac{\sigma^2}{n}$

$\hat{\beta}_{i1} \sim$ Normal mean $= \beta_{i1}$, var $= \frac{\sigma^2}{S_{xx}}$

7.3890

2 CEV chart [+5] +5/5

Let $X \sim N(\mu = 3, \sigma = 1)$ and $C = 2$. If $X > C$, the observation X is censored by C. Hence, the observed value is $\min(X, C)$.

(1) Find the probability of censoring $\Pr(X > C)$.

(2) Find the value of $E[X \mid X > C]$ (including mathematical derivation).

(3) We impute the observation by $X^{*} = X\mathbf{I}(X \le C) + w_c \mathbf{I}(X > C)$. Find the CEV w_c such that $E[X^{*}] = E[X]$ (including mathematical derivation). n=5

(4) Find the lower control limit for X^{*} using the Figure of Steiner & Mackay (2000).

(5) We obtained censored samples (1.5, 2, 2, 1, 2). Is this out-of-control?

(1)
$$P(X>C) = P\left(\frac{X-\mu}{\sigma} > \frac{C-\mu}{\sigma}\right) = 1-\Phi\left(\frac{C-\mu}{\sigma}\right) = 1-\Phi\left(\frac{2-3}{1}\right) = 1-\Phi(-1)$$
$$= 1-(1-\Phi(1)) = \Phi(1) = 0.84134$$

(2) $X \sim N(\mu=3, \sigma=1)$
$$f(X \mid X>C) = \frac{f(x)}{P(X>C)} I(x>C) = \frac{1}{P_c} f(x) I(x>C) \quad ; P_c = P(X>C) = P(Z>Z_c)$$
censored probability. $Z \sim N(0,1)$

$$\Rightarrow E(X \mid X>C) = \int_C^{\infty} \frac{1}{P_c} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{\frac{-(x-\mu)^2}{2\sigma^2}} dx \quad \text{Let } Z = \frac{X-\mu}{\sigma} \quad \therefore X = \mu + \sigma Z$$
Let $Z_c = \frac{C-\mu}{\sigma}$
$$= \int_{\frac{C-\mu}{\sigma}}^{\infty} \frac{1}{P_c} (\mu+\sigma z) \frac{1}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz = \frac{1}{P_c} \times \mu \times \underbrace{P(Z>Z_c)}_{P_c} + \int_{Z_c}^{\infty} \frac{\sigma}{P_c} \frac{z}{\sqrt{2\pi}} e^{\frac{-z^2}{2}} dz$$
$$= \mu + \frac{\sigma}{P_c} \frac{1}{\sqrt{2\pi}} \left(-e^{\frac{-z^2}{2}} \Big|_{Z_c}^{\infty}\right) = \mu + \frac{\sigma}{P_c}\left(\frac{1}{\sqrt{2\pi}} e^{\frac{-Z_c^2}{2}}\right) = \mu + \frac{\sigma\,\phi(Z_c)}{P_c}$$
$$\therefore E(X \mid X>C) = 3 + \frac{1}{\Phi(1)} \frac{1}{\sqrt{2\pi}} e^{\frac{-1}{2}} = 3.2876, \quad \text{where } Z_c = \frac{C-\mu}{\sigma} = -1$$

(3)
$$E(X^*) = E(X^* \mid X>C)\, P(X>C) + E(X^* \mid X<C)\, P(X<C)$$
$$= \underbrace{E(X^* \mid X>C)}_{W_c} P_c + E(X \mid X<C)(1-P_c)$$
$E(X^*) = E(X)$
$$E(X) = E(X \mid X>C)\, P(X>C) + E(X \mid X<C)\, P(X<C)$$
$$= E(X \mid X>C)\, P_c + E(X \mid X<C)(1-P_c)$$

$\therefore W_c = E(X \mid X>C)$
$= 3.2876$

$1-P_c = 1-0.84134 \doteq 0.16.$

(4)

$LCL = \mu + \sigma\, LCL_z$, use the figure $LCL_z = -1.15$ ($n=5$)

$= 3 + 1 \times (-1.15) = 3 - 1.15 = \underline{\underline{1.85}}$ ✓

(5)

$\bar{X} = \frac{1}{5}(1.5 + W_c + W_c + 1 + W_c) = \frac{1}{5}(2.5 + 3 \times 3.2876) = \underline{\underline{2.47256}} > LCL = 1.85$

✓

$\therefore$ In-Control.

+12/12

3 Matrix algebra [+3] +3/3

(1) Is $\Sigma = \begin{pmatrix} 4 & -2 \\ -4 & 2 \end{pmatrix}$ positive definite?

(2) Find the eigenvalues and eigenvectors of $\Sigma = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$. Here, eigenvectors must be standardized.

(3) Is $\Sigma = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$ positive definite?

(4) Write down the spectral decomposition of $\Sigma = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$.

(5) Find $\Sigma^{1/2}$ and check that $\Sigma^{1/2}\Sigma^{1/2} = \Sigma$.

(6) Find $\Sigma^{-1/2}$ and check that $\Sigma^{-1/2}\Sigma^{-1/2} = \Sigma^{-1}$.

(1) $|\Sigma - \lambda I| = \begin{vmatrix} 4-\lambda & -2 \\ -4 & 2-\lambda \end{vmatrix} = (\lambda-4)(\lambda-2)-8 = \lambda^2-6\lambda = \lambda(\lambda-6)$

$\therefore \lambda = 0$ or 6 if Σ is p.d, then $\lambda > 0$ eigenvalue positive, but Σ's eignvalue not all positive

Hence, Σ is not p.d.

(2)

$|\Sigma - \lambda I| = \begin{vmatrix} 5-\lambda & -4 \\ -4 & 5-\lambda \end{vmatrix} = (\lambda-5)^2-16 = \lambda^2-10\lambda+9 = (\lambda-1)(\lambda-9)$ $\therefore \lambda_1=1,\ \lambda_2=9$.

① $\lambda_1=1$ $\therefore$ By $\Sigma x = \lambda x \Rightarrow \begin{bmatrix} 4 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = x_2$ $\therefore V_1 = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

② $\lambda_2=9$ $\therefore \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_1 = -x_2 \Rightarrow V_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\Rightarrow e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. $\Rightarrow e_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Hence, eigenvalue : $\lambda_1=1, \lambda_2=9$ eigenvector : $e_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $e_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

(3) Since, $\lambda_1, \lambda_2 > 0$, Hence, Σ is p.d.

(4) $\Sigma = 1 \times \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}[1, 1] + 9 \times \frac{1}{2} \times \begin{bmatrix} -1 \\ 1 \end{bmatrix}[-1, 1]$. $\Sigma = \lambda_1 e_1 e_1' + \lambda_2 e_2 e_2'$

$= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + \frac{9}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 1\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} + 9\begin{bmatrix} \frac{1}{2} & \frac{-1}{2} \\ \frac{-1}{2} & \frac{+1}{2} \end{bmatrix}$.

(5) since Σ can be written as $\Sigma = \Gamma \Lambda \Gamma'$, $\Gamma\Gamma' = 1$.

$$\therefore \Sigma = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}'$$

$$\therefore \Sigma^{\frac{1}{2}} = \Gamma \Lambda^{\frac{1}{2}} \Gamma' = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & -3 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -2 & 4 \end{bmatrix} \times \frac{1}{2} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

check: $\Sigma^{\frac{1}{2}} \Sigma^{\frac{1}{2}} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix} = \Sigma$

(6)

$$\Sigma^{-\frac{1}{2}} = \Gamma \Lambda^{-\frac{1}{2}} \Gamma' = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} 1 & \frac{-1}{3} \\ 1 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} \frac{4}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{4}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

$$\Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}} = \frac{1}{9} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$$

而 $\Sigma^{-1} = \frac{1}{|\Sigma|} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{25-16} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} = \frac{1}{9} \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix}$.

$$\therefore \Sigma^{-1} = \Sigma^{-\frac{1}{2}} \Sigma^{-\frac{1}{2}}$$

Determining CEV Control Limits

An important question related to CEV weight control charts is how to choose appropriate control limits. The position of the appropriate control limits depends on both the sample size and the in-control probability censored. However, due to the effect of different degrees of censoring there is no generally applicable formula, such as the traditional plus or minus three standard deviation limits, that gives the appropriate control limits for CEV weight control charts. Figures 1 and 2 are provided simulation results to aid in the choice of control limits for the CEV $\overline{X}$ and S control charts. The figures are based on the assumption that the in-control proportion censored is known. Figure 1 gives the standardized lower control for the CEV $\overline{X}$ chart that has a theoretical false alarm rate of .0027. This particular false alarm rate was chosen to match the false alarm rate aimed for with the traditional Shewhart $\overline{X}$ control chart. Similarly Figure 2 gives the standardized upper control limit for a CEV S chart that yields a false alarm rate of .0027. Note that the horizontal axes in both Figures 1 and 2 are on a log scale.

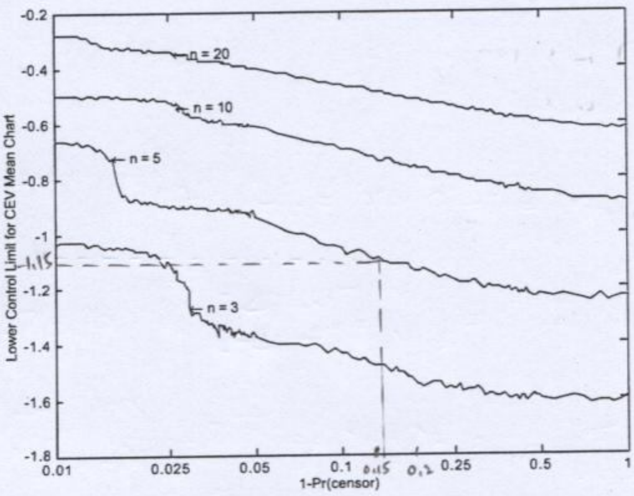

Figure 1: Plot of the Standardized Lower Control Limit for the CEV $\overline{X}$ chart