HW #3 Quality Control, Spring 2013, Due 6/7 (Fri),

1. [+5] Perform profile monitoring for the data in Table 10.3. The value of α is chosen so that the in-control ARL is 200 [p.479 of the textbook]. Then, compare your chart with Fig. 10.28. [provide all necessary information, such as UCL/LCL, center, calculation of α, regression estimates, etc.; missing information leads to the reduction in your score]

2. [+2] I told you in class

3. [+6] Let $\mathbf{X}_{1k},...,\mathbf{X}_{nk} \overset{iid}{\sim} N_3(\boldsymbol{\mu},\Sigma),\ k=1,..,m$, where $\boldsymbol{\mu}=\begin{pmatrix}-1\\-1\\-1\end{pmatrix}$ and $\Sigma=\begin{pmatrix}1 & 0.5 & 0.5\\0.5 & 1 & 0.5\\0.5 & 0.5 & 1\end{pmatrix}$.

Let $n = 25$, $m = 1000$, and $\alpha = 0.05$.

(1) [+1]Calculate Σ^{-1} (with derivation).

(2) [+2]Generate the data by R `mvrnorm` and draw a chi-square chart. Compare the observed ratio of out-of-control points with the nominal $\alpha = 0.05$.

(3) [+1]If $\boldsymbol{\mu}$ and Σ are unknown, we estimate them by $\overline{\overline{\mathbf{X}}}=\frac{1}{m}\sum_{k=1}^{m}\overline{\mathbf{X}}_k$ and $S=\frac{1}{m}\sum_{k=1}^{m}S_k$,

where $\overline{\mathbf{X}}_k=\frac{1}{n}\sum_{i=1}^{n}\mathbf{X}_{ik}$ and $S_k=\frac{1}{n-1}\sum_{i=1}^{n}(\mathbf{X}_{ik}-\overline{\mathbf{X}}_k)(\mathbf{X}_{ik}-\overline{\mathbf{X}}_k)'$. Compare $\overline{\overline{\mathbf{X}}}$ and S with $\boldsymbol{\mu}$ and Σ, respectively.

(4) [+2] Draw a chi-square chart by replacing $\boldsymbol{\mu}$ and Σ by $\overline{\overline{\mathbf{X}}}$ and S, respectively, and by using the same UCL and LCL as (2). Compare the observed ratio of out-of-control points with the nominal $\alpha = 0.05$. (This is called "the Hotelling T^2 chart")

HW #3 Quality Control

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1.

We have ARL = 200, σ = 0.06826, $\bar{x} = \frac{0.76+3.29+8.89}{3}$ = 4.3133 and model $y_{ij} = 0.2817 + 0.9767x_j + \varepsilon_{ij}$.

```
> # data
> x = c(0.76,3.29,8.89)
> y = c(1.12,3.49,9.11,0.99,3.53,8.89,1.05,
+           3.46,9.02,0.76,3.75,9.3,0.96,3.53,
+           9.05,1.03,3.52,9.02)
>
> # center data
> x_bar = mean(x)
> centered_x = x - x_bar
>
> # model: yij = 0.2817 + 0.9767xj + error
> 0.2817 + 0.9767 * x_bar
[1] 4.494533
> # centered model: yij = 4.494533 + 0.9767(xj - x_bar) + error
> beta = c(4.494533,0.9767)
```

Hence, the centered model is

$$y_{ij} = 4..494533 + 0.9767(x_j - \bar{x}) + \varepsilon_{ij}.$$

To draw the chart of β_0, β_1, σ^2 (MSE), we need to calculate $\widehat{\beta_{i0}}$, $\widehat{\beta_{i1}}$ and MSE_i.

```
> Sxx = sum(centered_x^2)
>
> # Calculate beta hat and MSE
> beta_hat = matrix(0,6,2)
> MSE = numeric(6)
> for(i in 1:6){
+     a = 1+3*(i-1)
+     b = 3*i
+     Y = y[a:b]
+     beta_hat[i,1] = mean(Y)
+     beta_hat[i,2] = sum(Y*centered_x)/Sxx
+     MSE[i] = sum((Y-beta_hat[i,1]-beta_hat[i,2]*centered_x)^2)
+ }
```

```
> beta_hat
             [,1]        [,2]
[1,] 4.573333 0.9862215
[2,] 4.470000 0.9692984
[3,] 4.510000 0.9823952
[4,] 4.603333 1.0406046
[5,] 4.513333 0.9935296
[6,] 4.523333 0.9826744
> MSE
[1] 8.627677e-03 4.234966e-03
[3] 3.137103e-03 7.032217e-02
[5] 1.750634e-03 8.097610e-06
```

Since ARL = 200, $\alpha^{overall} = 0.005$, $\alpha = 1 - (1 - \alpha^{overall})^{1/3} = 0.00167$ and $z_{\alpha/2} = 3.143492$.

```
> (alpha = 1-(1-0.005)^(1/3))
[1] 0.001669452
> sigma = 0.06826
> (z = qnorm(1-alpha/2))
[1] 3.143492
```

In addition, $UCL_{\beta 0} = \beta_0 + z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} = 4.6184$, $LCL_{\beta 0} = \beta_0 - z_{\alpha/2}\sqrt{\frac{\sigma^2}{n}} = 4.3706$,

$UCL_{\beta 1} = \beta_1 + z_{\alpha/2}\sqrt{\frac{\sigma^2}{S_{xx}}} = 1.013$, $UCL_{\beta 1} = \beta_1 - z_{\alpha/2}\sqrt{\frac{\sigma^2}{S_{xx}}} = 0.9402$,

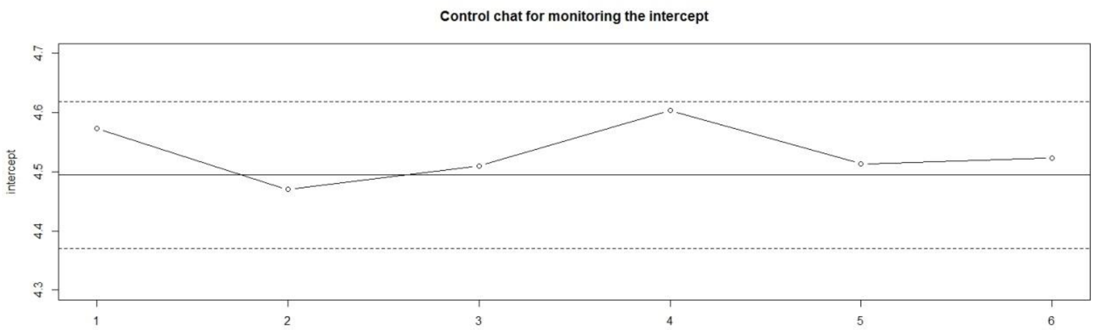
$UCL_{MSE} = 0.0520$ and $LCL_{MSE} = 5.0997 \times 10^{-9}$

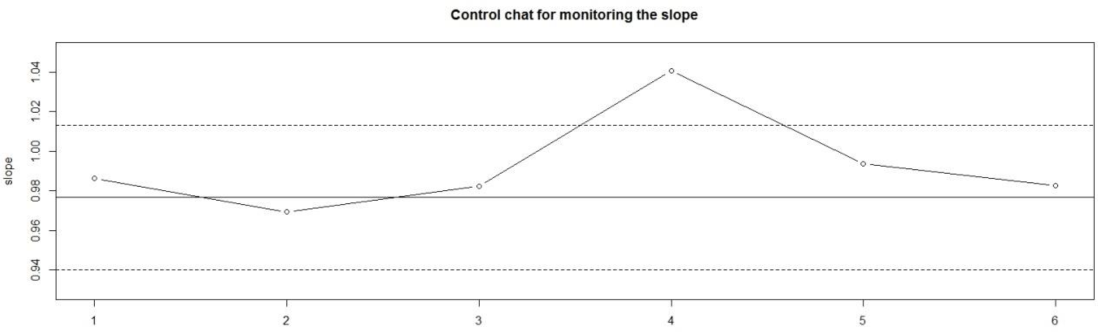
```
> se_beta = sigma * sqrt(c(1/3,1/Sxx))
>
> # imformation for chart for beta0
> U0 = beta[1]+z*se_beta[1]
> L0 = beta[1]-z*se_beta[1]
> cat("\nUCL = ",U0,"\nLCL = ",L0,
+        "\ncenter = ",beta[1],"\n\n")

UCL =  4.618418
LCL =  4.370648
center =  4.494533


>
> # imformation for chart for beta1
> U1 = beta[2]+z*se_beta[2]
> L1 = beta[2]-z*se_beta[2]
> cat("\nUCL = ",U1,"\nLCL = ",L1,
+        "\ncenter = ",beta[2],"\n\n")
```

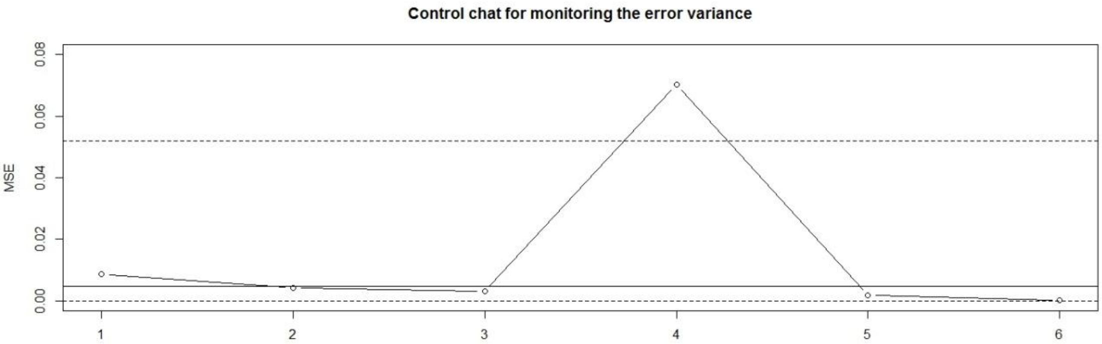
```
UCL =    1.013169
LCL =    0.9402314
center =    0.9767

>
> # imformation for chart for sigma2
> U = sigma^2*qchisq(1-alpha/2,1)
> L = sigma^2*qchisq(alpha/2,1)
> cat("\nUCL = ",U,"\nLCL = ",L,
+         "\ncenter = ",sigma^2,"\n\n")

UCL =    0.05201032
LCL =    5.099652e-09
center =    0.004659428
```

Next, plot the chart of β_0, β_1, σ^2 (MSE).

```
> plot(beta_hat[,1] , type = "b" ,
+ ylim = c(4.3,4.7) ,
+ xlab = "" , ylab = "intercept" ,
+ main = "Control chat for monitoring the intervept")
> abline(U0,0,lty = 2)
> abline(L0,0,lty = 2)
> abline(beta[1],0)
> plot(beta_hat[,2] , type = "b" ,
+ ylim = c(0.93,1.05) ,
+ xlab = "" , ylab = "slope" ,
+ main = "Control chat for monitoring the slope")
> abline(U1,0,lty = 2)
> abline(L1,0,lty = 2)
> abline(beta[2],0)
> plot(MSE , type = "b" ,
+           ylim = c(0,0.025) ,
+           xlab = "" , ylab = "MSE" ,
+           main = "Control chat for monitoring the error 
variance")
> abline(U,0,lty = 2)
> abline(L,0,lty = 2)
> abline(sigma^2,0)
```







These figures are pretty similar to the figures in textbook, except the UCL_{MSE} (it's close to 0.005 in textbook).

2.

(1) Prove $\mathbf{X} \sim N_p(\boldsymbol{\mu},\Sigma) \Rightarrow \mathbf{Z} = \Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu}) \sim N_p(0_p, I_p)$

Since $\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})$ is linear in $\mathbf{X}$, $\mathbf{Z}$ is multivatiate normal.

Moreover,

$$E[\mathbf{Z}] = E[\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})] = \Sigma^{-1/2}(E[\mathbf{X}] - \boldsymbol{\mu}) = 0_p$$

$$Cov[\Sigma^{-1/2}(\mathbf{X} - \boldsymbol{\mu})] = \Sigma^{-1/2}\, Cov[\mathbf{X} - \boldsymbol{\mu}]\, \Sigma^{-1/2'} = \Sigma^{-1/2}\, Cov[\mathbf{X}]\, \Sigma^{-1/2'}$$

$$= \Sigma^{-1/2}\, \Sigma^{1/2'}\, \Sigma^{1/2}\, \Sigma^{-1/2'} = I_p$$

Hence, $\mathbf{Z} \sim N_p(0_p, I_p)$.

(2) Prove $\mathbf{X}_1,\ldots,\mathbf{X}_n \sim N_p(\boldsymbol{\mu},\Sigma) \Rightarrow \overline{\boldsymbol{X}} \sim N_p(\boldsymbol{\mu},\frac{\Sigma}{n})$

Since $\overline{\boldsymbol{X}}$ is linear in $\mathbf{X}_1,\ldots,\mathbf{X}_n$, $\overline{\boldsymbol{X}}$ is multivariate normal.

Moreover,

$$E[\overline{\boldsymbol{X}}] = \frac{1}{n} E[\mathbf{X}_1+\ldots+\mathbf{X}_n] = \frac{1}{n}\sum_{i=1}^{n} E[\mathbf{X}_i] = \frac{1}{n}\sum_{i=1}^{n} \boldsymbol{\mu} = \boldsymbol{\mu}$$

$$Cov[\overline{\boldsymbol{X}}] = \frac{1}{n^2} Cov[\mathbf{X}_1+\ldots+\mathbf{X}_n] = \frac{1}{n^2}\sum_{i=1}^{n} Cov[\mathbf{X}_i] = \frac{1}{n^2}\sum_{i=1}^{n} \Sigma$$

$$= \frac{\Sigma}{n}$$

Hence, $\overline{\boldsymbol{X}} \sim N_p(\boldsymbol{\mu},\frac{\Sigma}{n})$.

3.

(1)

Since $\Sigma = (1-0.5)\, I_3 + 0.5\, \mathbf{11}'$,

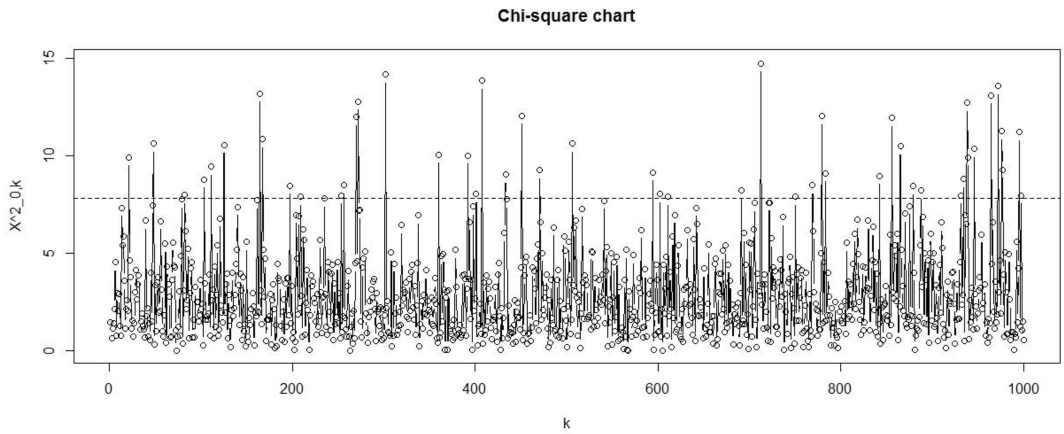
$$\Sigma^{-1} = \frac{1}{1-0.5} I_3 - \frac{0.5}{(1-0.5)[1+0.5(3-1)]} \mathbf{11}' = 2\, I_3 - 0.5\, \mathbf{11}'$$

$$= \begin{pmatrix} 1.5 & -0.5 & -0.5 \\ -0.5 & 1.5 & -0.5 \\ -0.5 & -0.5 & 1.5 \end{pmatrix}$$

(2)

```
library(MASS)
mu = c(-1,-1,-1)
sigma = 0.5*diag(1,3) + 0.5*rep(1,3)%*%t(rep(1,3))

> set.seed(465)
chi = numeric(1000)
for(i in 1:1000){
   X = mvrnorm(25,mu,sigma)
   X_bar = colMeans(X)
   temp = X_bar - mu
   chi[i] = 25*t(temp)%*%solve(sigma)%*%(temp)
}
```

```
UCL = qchisq(0.95,3)

U = max(max(chi),UCL)
plot(chi , type = "b" , ylim = c(0,U+0.1) ,
        main = "Chi-square chart" , xlab = "k" ,
        ylab = "X^2_0,k")
abline(UCL , 0 , lty = 2)
> (ratio = sum(chi > UCL)/1000)
[1] 0.049
```



The ratio of out-of-control points is 0.049 and it's close to the nominal α=0.05.

(3)

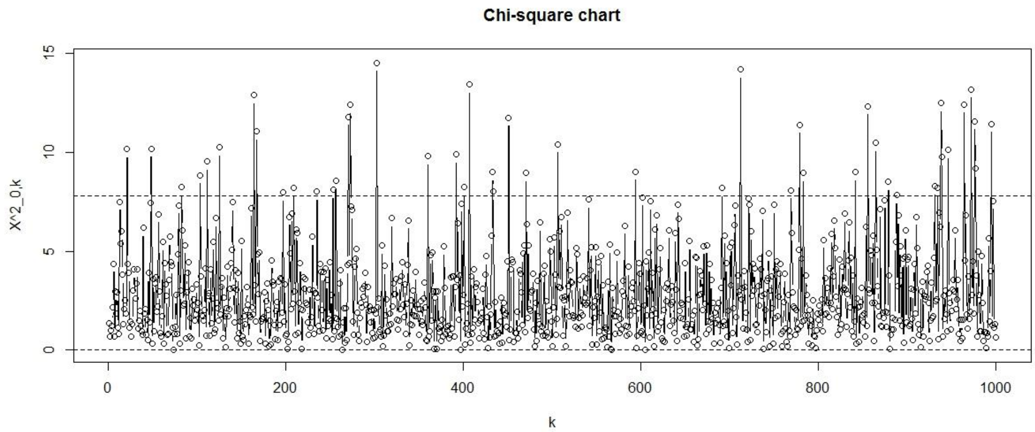
```
> library(MASS)
> mu = c(-1,-1,-1)
> sigma = 0.5*diag(1,3) + 0.5*rep(1,3)%*%t(rep(1,3))
>
> set.seed(465)
>
> X_bar_k = matrix(0,3,1000)
> Sk = array(0,dim=c(3,3,1000))
> for(i in 1:1000){
+     X = mvrnorm(25,mu,sigma)
+     X_bar_k[,i] = colMeans(X)
+     Sk[,,i] = cov(X)
+ }
>
> (X_bar_bar = rowMeans(X_bar_k))
[1] -1.013276 -1.010876 -1.005413
>
> (S = rowMeans(Sk , dim = 2))
```

```
                 [,1]      [,2]      [,3]
[1,] 1.0244434 0.5029764 0.5072474
[2,] 0.5029764 1.0011144 0.5024538
[3,] 0.5072474 0.5024538 1.0066791
```

The $\bar{\bar{X}}$ and S seem to be close to the **μ** and Σ.

(4)

```
> chisq = numeric(1000)
> for(j in 1:1000){
+     temp = X_bar_k[,j] - X_bar_bar
+     chisq[j] = 25*t(temp)%*%solve(S)%*%temp
+ }
>
> UCL = qchisq(0.95,3)
>
> U = max(max(chisq),UCL)
> plot(chisq , type = "b" , ylim = c(0,U+0.1) ,
+         main = "Chi-square chart" , xlab = "k" ,
+         ylab = "X^2_0,k")
> abline(UCL , 0 , lty = 2)
> abline(0 , 0 , lty = 2)
>
> (ratio = sum(chisq > UCL)/1000)
[1] 0.046
```



The ratio of out-of-control points is 0.046. It's close to the nominal α=0.05, but slightly less than the value in (2).